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APPLIED MATHEMATICAL MODULES FOR USE IN A LINEAR ALGEBRA SERVICE COURSE

SHIRLEY JO ZANDER

A Dissertation Submitted in Partial
Fulfillment of the Requirements
For the Degree Of

DOCTOR OF ARTS

Department of Mathematics

ILLINOIS STATE UNIVERSITY

1990

DISSERTATION APPROVED:

4/24/90

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4/24/90

Date Michael Plantholt

4/24/90

Date Robert Ritt

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272 Pages May, 1990

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Each application was designed so that and instructor of linear algebra could use it either as and independent study project or as an integrated part of the course. These three applications are arranged by degree of difficulty and sequenced as they would be introduced in a linear algebra course.

On the whole, the students participating in the testing of these applications felt that exposure to examples such as these during their Mathematics 175 course would have increased their understanding and enjoyment of linear algebra. This finding supports the overall goal of this study. (KR)

APPLIED MATHEMATICAL MODULES FOR USE IN A
LINEAR ALGEBRA SERVICE COURSE

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I especially wish to thank my mother, Jane Zander, for her constant support, patience, and the encouragement she has given me throughout the years spent in graduate school.

S. J. Z.

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CHAPTER I

INTRODUCTION

The Need for Applications in a Linear Algebra Service Course

Linear algebra is an area of mathematics which can be applied to a wide variety of other disciplines with both scientific and non-scientific orientations. Each year new applications are being discovered for this very versatile subject. Because linear algebra and its applications are used in many disciplines other than mathematics, it has, in many instances, become a requirement for non-mathematics majors to take a course in linear algebra. Theory developed in other fields of study may require the use of only a very limited selection of concepts from linear algebra, and the faculty in these fields usually prefer to teach the theory involving the concepts from linear algebra as an integrated part of one of their own courses. However, it is important that students understand the basics of linear algebra before applying the specialized concepts in linear algebra to their particular field of study. A strong background can be obtained by taking a service course in linear algebra which will provide students with the concepts necessary to understand the specific applications found in their chosen fields and subsequently apply this knowledge to real world situations. This conclusion is supported by the PRIME-80 Conference (1978) which strongly recommended that an appropriate balance between applications and fundamental principles be maintained in any course in mathematics. Also, the conference concluded that the structure provided by the fundamental principles will endure beyond any changes that a particular application may undergo (p. 12).

The purpose of this study is to develop those applications which can be used to supplement an undergraduate course in linear algebra for non-mathematics majors. One of the serious difficulties in teaching this particular type of undergraduate course is the students' lack of ability to see the "big picture." They simply have not had enough experience in their major course of study to see how and where linear algebra can be applied. They also lack an appreciation of the value, in their discipline, of having a solid background in linear algebra.

A linear algebra service course provides the necessary mathematical skills to understand an application involving linear algebra. In addition, Bell (1980) claims that a course with greater emphasis on applications would give the students a better appreciation for the role of mathematics in our civilization as a whole. He maintains that

> . . . without this emphasis, it is as if we were spending long years explaining the inner workings of an intricate and powerful machine without ever showing how to use it. (p. vi)

It has been recommended (CUPM, 1983, p. 5) that mathematics courses be taught in a manner so as to create interest and enthusiasm for mathematics by students who are not mathematics majors. This is a tall order, because many times students view the required linear algebra course as "just another math course that has to be taken." In an earlier work, the CUPM (1972) panel on applied mathematics stated that

> . . . many students lose their enthusiasm for mathematics even as a tool because their mathematics courses seem unrelated to their own discipline . . . , the best way to demonstrate the power and utility of mathematical ideas to these students and thereby to sustain their interest is to introduce applications to other fields. (p. 11)

It is, indeed, the rare student who can be told that linear algebra will be valuable in future course work, and then works hard to understand the concepts without being given examples of how and where it will be used. On the other hand, because of difficulties in scheduling classes, many students take the linear algebra service course out of sequence and subsequently realize how valuable it would have been to have mastered the skills from linear algebra before the applications were discussed in their major courses. Thus, the students taking a linear algebra service course come from many diverse fields of study, and their motivations toward linear algebra are just as varied. Most students find it is easier to understand algebraic concepts when concrete applications are used to illustrate and motivate these concepts (Dornhoff & Hohn, 1978, p. v). Keeping this in mind, it is imperative to assemble as many applied problems as possible, from different fields of study which can be solved using linear algebra. These applications could then be introduced throughout the course to motivate students and to give them a taste of where they will find uses for linear algebra in their fields of study as well as subsequent employment. The abundance of applications will also make students aware that seemingly unusable theoretical ideas from the classroom can readily be applied to a broad spectrum of real world situations.

Ames (1980) describes the changes which take place when applications are introduced into the classroom situation:

> Bringing the real world into my mathematics classes has been at times dangerous or troublesome. Dangerous because it raises the expectations of the students (and myself), creates emotional as well as intellectual involvement, demands conceptual understanding while also demanding computational skill . . . , draws on mathematical skills not "in this unit," puts us on unsure ground, leads to murky waters, creates tangential interests, and devours time. Troublesome because it makes us ask for more, desire deeper understandings, see the world differently, ask increasingly more difficult questions about the world, and realize what a long way we've got to go to "get educated." So why try to deal with applications of mathematics within my classroom? Because it's exciting and invigorating; it develops mathematical power . . . ; it seems to create real intellectual growth; it has a high immediate impact on students and a long term residual effect; it makes one want to understand and look for "whys" as well as "hows." (p. 10)

Thus Ames feels that it is possible to create excitement and interest towards mathematics through the use of applications. This is the goal which this study endeavors to accomplish.

Use of Applications in the Classroom

Applications of topics in the form of modules are very flexible in that they can be used to enhance the students' understanding and enrich their appreciation of the subject. They provide a means by which the student can see the steps that are necessary to transform a real-life situation into a mathematical problem which can be solved by using techniques from linear algebra. This is especially important for students who do not view mathematics as being a subject which can be directly applied to problems in everyday life. The following paragraphs describe ways in which these applications might be effectively used by an instructor.

The first few days of a course are critical. This is the time to initiate the students' interest in linear algebra. Once the instructor has been able to establish in the students' minds that linear algebra is an important subject which can be applied to many other fields in addition to mathematics, the students won't be so inclined to dismiss the theory as having no practical value. One way to reinforce the theory is to have a large selection of applications so that it is possible to present those which best match the interests of the students in the course. Even though it is essential for students to see a variety of applications, it is also important that some applications be selected for in-depth study (CUPM, 1983, p. 49). In any basic linear algebra course, the first topic which is usually discussed is systems of linear equations. Thus, any set of applications should include ones which are basic enough to be presented during the first few days of class. The general ideas found in the student's first application would show how a real-life situation is interpreted, and then transformed into a linear algebra problem which is easily solvable using a linear system of equations. Subsequent applications should be developed to integrate skills and topics that the students learn as they progress through a course in linear algebra. These applications need to cover a wide variety of topics and incorporate various levels of skills as they are learned.

Applications can be incorporated into the course by having students work individually or in small groups on a particular application. After the individual or group has finished studying the application, it could be presented to the class for discussion. This would help build student involvement, a basic goal of presenting applications in the classroom (Ames, 1980, p. 12). An advantage of group and individual presentations of problems is that more intricate applications could be used. Since each application is presented to the class, students will be exposed to a more diverse variety of applications. Also, an instructor may present an application as part of class and assign homework from the lecture material. Alternatively, an instructor may present the first part of the application and allow the students to complete the remainder of the presentation as homework. In addition, applications may also be assigned as enrichment-type homework, allowing students to work through the application at their own pace. No matter how applications are introduced to students, it is important to have an extensive collection available to allow students to choose ones which will both interest and challenge them.

Design of the Applications

This study encompasses the development and field testing of three applications designed for implementation in a variety of ways by an instructor of a linear algebra course for non-mathematics majors. Each application module was developed to be completed as an independent project by the student. Each application contains a list of required prerequisite knowledge in linear algebra which should be covered in class prior to beginning the application, as well as any background information needed from the particular field of study. The three applications in this study are from the fields of biology, graph theory, and physics. The biology application examines how a system of linear equations is used to determine the concentration of substances dissolved in a colored liquid. In the graph theory application, linear algebra is used to analyze properties of spanning trees and illustrate how they are employed to solve transportation problems. The third application shows how linear algebra is used to solve systems of second order linear differential equations which can be used to model small vibrations in molecules.

Chapter II provides information about the field testing of the three applications. The applications were field tested using students who voluntarily elected to take an independent study course during the spring semester of 1990 at Illinois State University. Chapter II also includes academic background information on the participating students, their comments on each application and the resulting revisions. Chapter III summarizes the approach used to create these applications and the subsequent field testing, as well as the process used to revise them. The three applications are found in the appendices.

CHAPTER II

ANALYSIS OF DATA FROM THE USE OF THE APPLICATIONS

Background of Participating Students

Following their development, three linear algebra applications were field tested to insure both student comprehension and educational value. Before field testing began, volunteers were sought in the fall semester of 1990 to participate in an independent study course. These volunteers were solicited from courses with students who had already completed Mathematics 175 (Linear Algebra), a sophomore level course offered by the Mathematics Department at Illinois State University. By drawing from a wide range of courses, it was hoped to obtain a broad cross section of students, especially those who were not majoring in mathematics. Although Illinois State University does not offer a linear algebra service course specifically designed for non-mathematics majors, Mathematics 175 is the course which is most similar to a linear algebra service course. Since these applications were designed to supplement a linear algebra service course, it was desired that Mathematics 175 be completed by some students during the previous semester while the remainder of the students had completed it at an earlier time.

The eight students in the course consisted of three graduate students and five undergraduates. Of the three graduate students, two were in mathematics and the third in chemistry. The undergraduate group consisted of two students in mathematics, one student with a double major in mathematics and finance, one in mathematics education, and one in chemistry. One of the mathematics graduate student had already completed Mathematics 337 (Linear Algebra), an upper level course. Mathematics 175 had just been completed by four of the students. The remaining three students, two graduate and one undergraduate, had completed Mathematics 175 more than a year earlier. Table 2 summarizes the pertinent information obtained from the Student Background Survey (Table 1) which the students completed on the first day of class.

TABLE 1

STUDENT BACKGROUND SURVEY

Name ____________________________________.

Year in college (circle one)

Freshman Sophomore Junior Senior Graduate Student

What is your major field of study? ___________________________________.

What mathematics courses have you taken at Illinois State University?

What semester and year did you take Math 175 (Linear Algebra)?

Semester____________ Year _______

Did you see any applications in your linear algebra course?

In other mathematics courses?

If so, do you remember what they were?

Have you ever taken an Independent Study course before?

TABLE 2

SUMMARY OF THE STUDENT BACKGROUND SURVEY INFORMATION

Year in College/ Field of Study	Semester Year of Math 175	Previous Mathematics Courses taken at Illinois State University (ISU)
Graduate/ Mathematics	Spring 1987	BS from ISU Math 363 (Graph Theory) Math 368 (Numerical Matrix Methods)
Graduate/ Mathematics	Fall 1987	BS from ISU Math 368 (Numerical Matrix Methods) Math 337 (Linear Algebra)
Graduate/ Chemistry	Fall 1987	Math 350 (Mathematical Statistics I) Math 340 (Differential Equations I) Math 260 (Introduction to Discrete Mathematics) Math 390 (Independent Study in Graph Theory)
Senior/ Mathematics and Finance	Fall 1988	Math 120-121 (Finite Mathematics for Business and Social Sciences) Math 145-147 (Calculus I, II and III) Math 340 (Differential Equations I) Math 350 (Mathematical Statistics I)
Senior/ Mathematics	Fall 1989	Math 210 (Symbolic Logic I) Math 236 (Introduction to Abstract Algebra I)
Senior/ Chemistry	Fall 1989	Math 147 (Calculus III)
Junior/ Mathematics Education	Fall 1989	Math 146-147 (Calculus II and III)
Junior/ Mathematics	Fall 1989	Math 145-147 (Calculus I, II and III) Math 164 (FORTRAN Programming) Math 385.01 (Actuarial Examination Preparation I)

Course Description

A syllabus, describing the material to be completed by each lesson along with homework assignments, was given to the students during the first class period. Class participation was required for all students except the two mathematics graduate students. These two students were directed to attempt each application without any interaction with the instructor or other students. This served as an independent evaluation of the application to see if the material would be interpreted as intended. The students who attended class either received a lecture on prerequisite material for the application or participated in a question and answer period. These lectures did not cover the material from the application but only reviewed linear algebra techniques which students had seen in Mathematics 175. This was especially helpful to those students who had completed this course more than a semester earlier, since some of the techniques had been forgotten. However, if a student asked a question concerning a particular concept from an application, the concept along with related ideas and techniques were openly discussed during class.

One of the goals of field testing is to insure that exercises are of the right length and difficulty. Field testing should also verify that the exercises reinforce the theory found in each application. Thus, it was important that each student correctly complete all exercises. To this end, the students were allowed to resubmit their homework until each exercise was correct. In actuality, relatively few exercises had to be resubmitted by the students.

Student Evaluation of Applications

A personal interview to discuss the essentials of the application was required after each student had completed all the exercises in the application. This interview also covered the student's feelings on the suitability and content of the applications in applying their linear algebra skills to the three specific situations. The form, shown in Table 3, was the one used to collect information from each student during their personal interviews.

TABLE 3

STUDENT INTERVIEW FORM

Application I, II, III. Name of Student ________________.

A. Application Content

1. Was the application readable?

2. Did you feel the material was too difficult? If so, which section?

3. Did you find the application challenging?

4. Did you enjoy this application?
   What did you like about it?
   What did you dislike about it?

B. Application Clarity and Comprehension

1. Were there any typographical errors?

2. What sections were unclear or difficult to understand?

3. Did the reading provide enough background information in the biology, graph theory, and small vibration theory to properly understand the material? If not, where does extra material need to be added?

4. Were there technical terms which need additional explanation?

5. Did the exercises help you understand the theory?

6. Did you read the material more than once before understanding the subject matter?

C. Exercise Content

1. Did you read the material before starting the exercises?

2. Were the exercises too easy? difficult? short? long?

D. Would you find this application beneficial as part of a linear algebra course?

We begin by summarizing the comments from the students obtained after they had completed the application of linear algebra to biology. All the students responded that they felt the application was readable and not difficult. The first application was designed to be presented at the beginning of a linear algebra service course. Since the students participating in the field testing had already completed a linear algebra course, they felt the application was not significantly challenging. Thus, the students found the exercises relatively easy to complete. In fact, all the students said they enjoyed the content of this application for they could easily see where the concepts of linear algebra were employed. The students also felt that none of the sections were too difficult nor unclear. They indicated that adequate background information was contained in the application for them to comprehend the material. Everyone felt that this application on biology would benefit students in a linear algebra course such as Mathematics 175.

Next, we summarize the students' comments from the graph theory application. All students indicated that the application was easily readable and mildly challenging. The linear algebra skills required for this application were not difficult. Since all students had mastered the use of these skills, they enjoyed graph theory and its real world application. The one area that students found challenging was the Improvement Method. This was understandable since the Improvement Method was the heart of the application and the primary technique used to solve transportation problems. The exercises were found to be of adequate length and quantity while being sufficient to clarify the theory. The students unanimously concluded that this application would be beneficial for students studying linear algebra.

Finally, we look at the students' comments on the application involving small vibration theory. This application was designed to be the most challenging as well as the one that required the most skill in linear algebra. In a course like Mathematics 175, many of the required skills are taught at the end of the course. Thus, these skills are usually the weakest and most easily forgotten by the student. That is why many of the students found this application to be the most difficult. Additionally, a course in differential equations would be helpful, but it is not required since the application contains an appendix showing how to solve the basic differential equations which were found in the application. Two of the students had not taken a course in differential equations and one student was taking it concurrently. These particular students required an extra amount of work and guidance in order to understand and successfully complete this application. From the Student Interview Form, all students described this application as challenging. Each student required various degrees of assistance in different areas due to their varied backgrounds in mathematics and diverse exposures to physics. The students felt that the clarity of the application and its comprehension were sufficient to provide adequate insight into the subject matter. The students indicated that the area in Section 2 which described vibrational modes needed to be supplemented. Because the material in the application was new to many of the students, the first two sections had to be read several times to insure proper understanding. However, because of this extra effort, it was much easier for the students to understand the third section, which was the primary objective of this application. A few of the students felt that some of the exercises were too long, but they also said that they needed to be of sufficient length as there was no way the exercises could be made shorter and still adequately convey the theory. The majority of the students indicated that this application might be more applicable to students who are taking Mathematics 337 rather than students in Mathematics 175. However, if it were included in Mathematics 175, students suggested that the application should be supplemented with class lectures discussing the material.

Revisions Made to the Applications

Following the field testing of these applications and in conjunction with inputs from the students, typographical errors were corrected, minor grammatical inconsistencies were remedied and some phrases were rearranged to improve clarity. As field testing progressed, it was found that supplemental information was needed. This information has now been included in the applications and is summarized below. (The revised applications appear in the appendices.)

Application I Linear Algebra applied to Biology

No significant changes were made to this application.

Application II Linear Algebra applied to Graph Theory

1. All diagrams which illustrate transportation problems (Figures 6.3-6.10 and A.10-A.27) were redrawn and supplemented to more clearly depict the information from the problem.
2. In Exercise 6.2 and 6.3 a statement was added requiring the students to include the objective function and constraint equations as part of their solution.

3. More details and diagrams were added to the solution of Exercise 6.3 to assist the instructor in correcting the student's homework.

Application III Linear Algebra applied to Physics

1. Figure 2.3 in Section 2 illustrates a spring-weight system with two blocks. This spring-weight system is modeled by the system of differential equations $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$. The theory that describes how to solve this system of differential equations was originally written using the system $\ddot{\vec{X}} = A \vec{X}$, and then the constant $\frac{k}{m}$ was reintroduced just before the final solution was found. Initially, an exercise was provided to have students verify the reintroduction of $\frac{k}{m}$. This method was very confusing to the students and as a consequence this section was rewritten showing how the theory could just have easily been done with the coefficient $\frac{k}{m}$ remaining. As a result, the exercise was deleted.
2. As indicated earlier in the Student Evaluation of the Applications section, the description of the modes of vibration with two blocks was difficult for students to understand. As a result, this section was rewritten to improve the conceptualization and to clarify the technique used to determine the vibrational modes.

CHAPTER III

SUMMARY

Summary of Findings

The purpose of this study was to develop several applications which could be used to supplement an undergraduate course in linear algebra for non-mathematics majors. These applications were drawn from the fields of biology, graph theory, and physics. The biology application examined how a system of linear equations could be used to determine the concentration of substances dissolved in a colored liquid. In the graph theory application, linear algebra was applied to analyze properties of spanning trees and to illustrate how they would be employed to solve transportation problems. The third application showed how linear algebra is used to solve systems of second order linear differential equations, which could be used to model small vibrations in molecules.

Each application was designed so that and instructor of linear algebra could use it either as an independent study project or as an integrated part of the course. These three applications are arranged by degree of difficulty and sequenced as they would be introduced in a linear algebra course.

As a result of field testing these applications, it was found that students gained a better appreciation for the fact that linear algebra could be used to model "real world" problems outside the field of mathematics. Of particular note, the chemistry graduate student who participated in the field testing indicated that the applications provided a better appreciation of linear algebra. The student also indicated that these applications provided a better insight into areas they had previously studied.

On the whole, the students participating in the testing of these applications felt that exposure to examples such as these during their Mathematics 175 course would have increased their understanding and enjoyment of linear algebra. This finding supports the overall goal of this study.

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APPENDIX A

Application I

Contents of Application I

Application I

Linear Algebra Applied to Biology

The Use of the Spectrophotometer to Determine the Concentration of a Colored Liquid

Linear Algebra Prerequisites: Being able to solve systems of linear equations using matrices.

Prerequisite Knowledge in Biology: None.

Section I Concentration of a Colored Liquid

The color of an object in white light is determined by the amount of light or visible electromagnetic radiation that it absorbs. White light is used, because it contains wavelengths from the entire spectrum. The visible range of the electromagnetic spectrum lies between 400 and 750 nanometers (nm). Thus, to see a color with the naked eye, the object must absorb light within this range. Many liquids are naturally colored and some clear liquids can be colored by adding certain reagents which do not change the concentration of the substance. The intensity of color depends on the amount of the substance present in the liquid which absorbs light. The more light at a particular wavelength which is absorbed by the liquid, the more intense the color becomes. To understand how the intensity of color can be measured, let us consider the following example. Suppose we have a vial of colored liquid illuminated by a beam of light. At a given wavelength some of the light that enters the vial will be absorbed and some will leave the vial. The component that leaves the vial is what we are considering to be "reflected". The spectra of the liquid is a graph of either the reflection or the absorption by the liquid compared to the wavelengths of white light. The spectra of the liquid describing the amount of radiation reflected is a graph with wavelength along the x-axis and a scale to show the reflection on the y-axis. Figure 1.1 shows an example of this type of spectra.

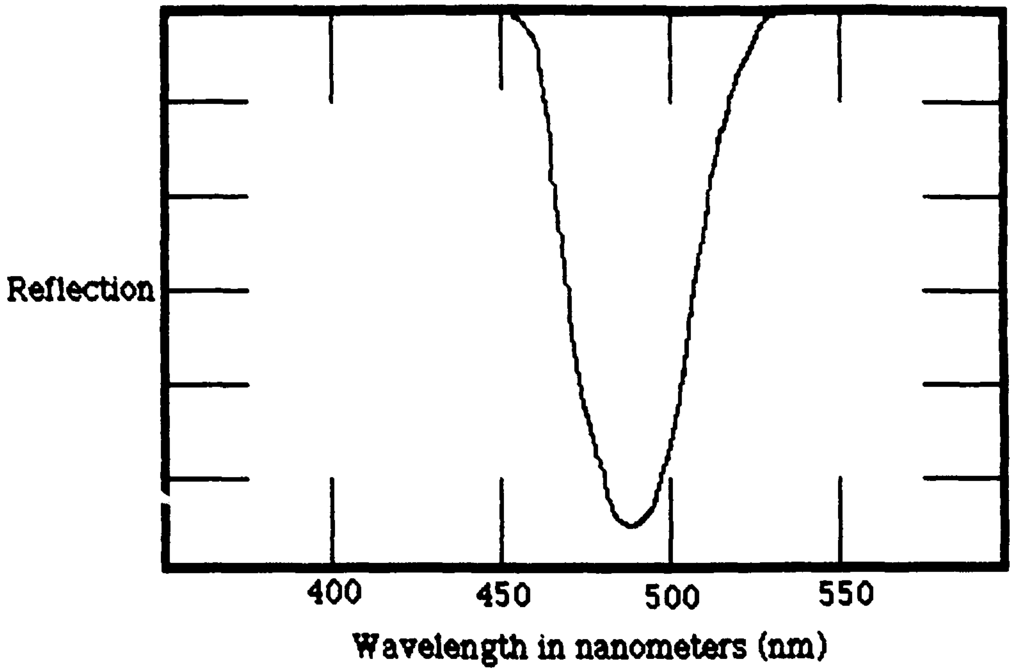


Figure 1.1

The graph of the spectra of the liquid describing the amount of radiation absorbed is very similar except the y-axis contains a scale to show the absorption or the amount of light that does not get through the liquid. Figure 1.2 shows an example of this type of spectra.

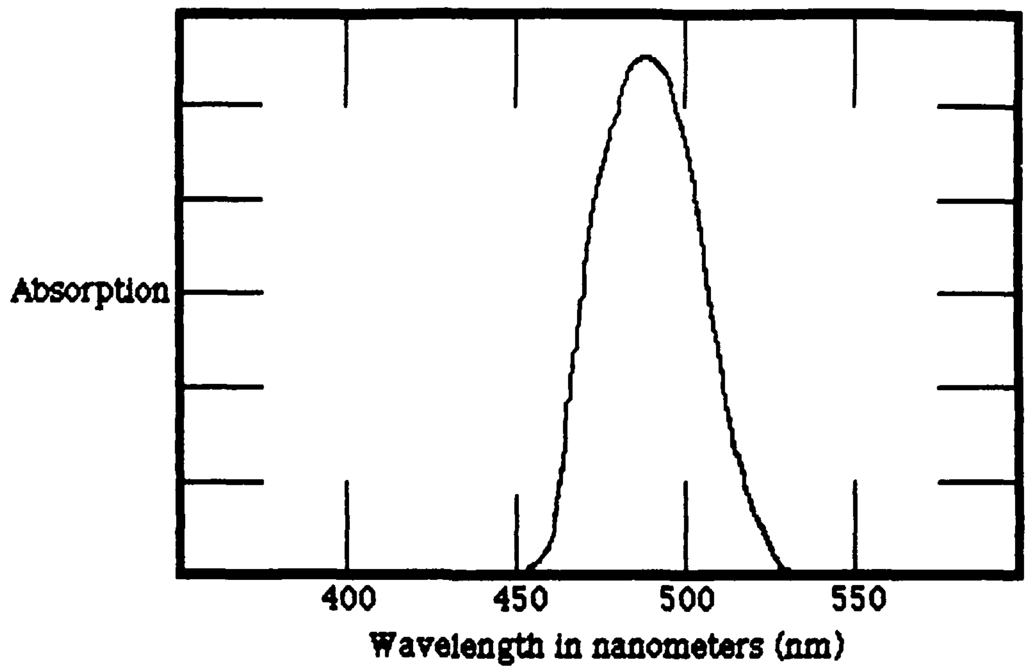


Figure 1.2

If the graphs in Figures 1.1 and 1.2 are spectra of the same liquid, they would be compliments of each other. For the remainder of this study, we will use the absorption type of spectra.

Colorimetry is an optical method used to determine the concentration of a colored liquid by comparing the intensity of its color with the intensity of a sample of the same liquid whose concentration is known. We call the sample of known concentration a **standard**. Thus, we will limit our investigation to liquids which are colored or can be colored by adding a reagent which reacts with the substance dissolved in the liquid, but does not affect the concentration and which forms a colored compound whose concentration can be determined colorimetrically. Additionally, we will only be interested in liquids whose intensity of color is linearly proportional to the concentration of the liquid. This means there is a constant of proportionality k, such that

$$(1.1) \qquad k * \text{concentration} = \text{intensity}.$$

The constant of proportionality depends on the substance in the liquid.

The most simplistic colorimetric method uses the naked eye to compare the intensity of color of a sample to the intensity of color of a standard while both are sitting side-by-side. This method is accurate only if both the sample and the standard are identical substances except for concentrations. That is, there are the same impurities in one solution as in the other. Also, the human eye interprets light which does not contain all of the wavelengths of the spectrum as white light. This causes distortions in color, because the eye can not compare two spectra which have gaps of missing wavelengths in the region where absorption occurs. We need to be very careful to avoid these situations so errors are not introduced into our measurement techniques when using this colorimetric method. According to Snell and Snell (1958, p. 1), this method was first used on iron and cobalt solutions (which are naturally colored) more than a century ago. Since then, there has been a tremendous improvement in the accuracy with which we are able to measure the intensity of colors. These advances are largely due to the invention of the **photoelectric cell.** This instrument interprets variations in intensity as changes in an electric current. A **colorimeter** measures the intensity of color of a liquid by shining a beam of white light through a filter that eliminates all but a specific band of wavelengths which then passes through the colored liquid. The amount of light not absorbed by the liquid (the light which passes through) is measured by a photoelectric cell. Using a filter to separate the spectrum causes the bands of wavelengths to be relatively wide which can easily cause an error in calculations. Also, if a change in wavelength is desired, a different filter must be used. Although there are more sophisticated devices now available, colorimeters are relatively portable and can be used for testing liquids in non-laboratory situations.

Many liquids appear clear to the naked eye, but when viewed in the ultraviolet or infrared spectrum (neither of which the human eye can see) have very distinctive absorptions. Thus, we are interested in an apparatus called a **Spectrophotometer**, which is capable of determining the changes in intensity of light of different wavelengths in the visible spectrum as well as in the ultraviolet and infrared spectrums. The spectrophotometer can be used to determine the concentration of liquids whose absorption wavelength falls in a wider range than liquids which can be measured by a colorimeter. Another advantage of using the spectrophotometric method compared to chemical analysis is that the procedure requires only a small quantity of solution to be tested. This can be important if the substance is expensive or difficult to obtain in a large quantity. Also, the solution often remains unchanged by the procedure, and subsequently can be used for other purposes. If the substance as well as the concentration is unknown, a spectrophotometer can be used to first determine the substance in the liquid, and then used to determine the concentration. To do this we first dissolve the unknown substance in a liquid and determine its absorption wavelengths. Then we compare these values to the absorption wavelengths of the liquid in which the unknown substance was dissolved, which is called a **blank.** A blank's concentration value is always zero. From this comparison we obtain the absorption wavelengths of the unknown substance which can be used to identify the substance. The spectrophotometer then uses a narrow band of the spectrum around the wavelength where the colored liquid absorbs radiation, to find its concentration. To understand how we can use the amount of radiation absorbed by the liquid to determine the concentration of the substance, we need to understand the physical laws that govern the absorption of light.

The fundamental equation we will use is the combination of two laws, **Beer's law** (the light absorbed by a layer of solution is directly proportional to the concentration of the colored substance) and **Lambert's law** (the light absorbed is directly proportional to the thickness of the solution). The equation which expresses the **Lambert-Beer law** is

$$\text{optical density}_{\lambda} = \log_{10} \frac{I_0}{I} = k\, c\, l, \tag{1.2}$$

where λ = the absorbing wavelength of the substance.
units: nanometers ($1\ \text{nm} = 10^{-7}\ \text{cm}$)

I_0 = intensity of the light at wavelength λ entering the sample.

(This is the same as the intensity of the light passing through the blank, where the concentration is zero.)

I = intensity of the light at wavelength λ exiting the colored liquid of unknown concentration.

k = specific extinction coefficient of the absorbing substance at λ.

This constant comes from the characteristic absorption coefficient for the liquid whose concentration we are trying to find.

units: $\frac{\text{liters}}{\text{mole cm}}$

c = concentration of liquid

units: $\frac{\text{moles}}{\text{liter}}$

l = the width of the solution that the transmitted light must pass through.

units: centimeters (cm)

Since the values for I_0 and I are measured in the same units, the ratio $\frac{I_0}{I}$ is dimensionless, which allows us to take the logarithm of the ratio. Most spectrophotometers are designed so that $\log_{10} \frac{I_0}{I}$, or the **optical density** at λ, can be read directly off a scale on the machine. When the concentration (c) is measured in moles per liter and the thickness (l) of the absorbing solution is one centimeter, we use the symbol ϵ_λ to represent the molar extinction coefficient of the substance at the wavelength λ. The units of ϵ_λ are $\frac{\text{liters cm}}{\text{mole cm}} = \frac{\text{liters}}{\text{mole}}$. Thus, the equation for optical density at λ becomes

$$\text{optical density}_\lambda = \log_{10} \frac{I_0}{I} = \epsilon_\lambda c. \qquad (1.3)$$

Since the optical density at λ is a dimensionless quantity, it is a good idea to check to make sure the units on the right side of the Equation (1.3) cancel out. Substituting in the appropriate units, we obtain

$$\text{(1.4)} \qquad \text{optical density}_\lambda = \epsilon_\lambda \, c = \frac{\text{liters}}{\text{mole}} \frac{\text{moles}}{\text{liter}} .$$

Our check is complete since all of the units on the right side cancel.

Figure 1.3 shows the basic set up of a spectrophotometer.

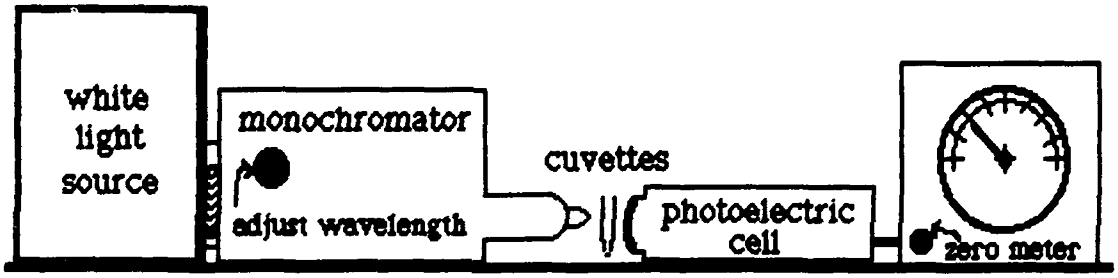


Figure 1.3

The monochromator allows us to select a narrow band of wavelengths around λ, the absorbing wavelength of the liquid, from the beam of white light. The narrow band of light can then be directed at either of two cuvettes. The cuvettes are a pair of matched glass vials whose front and back pieces are carefully constructed to be flat, parallel, and have the same thickness throughout. The two clear glass sides of the cuvettes must be kept clean and free of finger prints so the beam of light is not distorted. To remind the experimenter which of the sides are made out of the special glass, the other sides are frosted. The thickness of the cuvette is exactly one centimeter. Figure 1.4 shows an example of one type of cuvette.

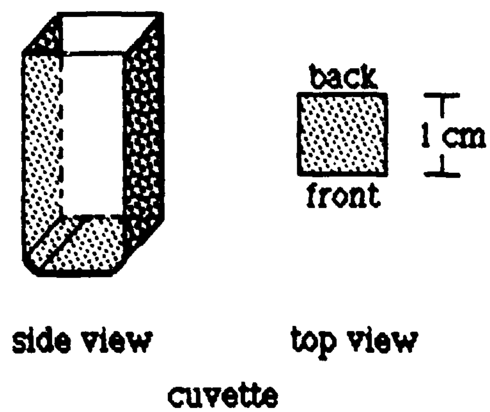


Figure 1.4

One of the two cuvettes placed in the spectrophotometer contains the liquid of unknown concentration and the other contains the blank. The beam of light from the monochromator alternates between the two cuvettes and the intensity of light which passes through each of the two solutions can be measured by the photoelectric cell. The optical density can be read directly off a meter on the spectrophotometer.

The following procedure is designed to find the concentration of a colored liquid containing only one known substance. Related steps have been grouped into sections.

A.1 Fill the cuvette with water (this is the blank) and place it in the spectrophotometer.

A.2 Adjust the spectrophotometer until the optical density reads zero.

B.1 Replace the cuvette containing water with one containing the standard dissolved in water.

B.2 Record the optical density of the standard.

C.1 Place the sample of liquid of unknown concentration in the spectrophotometer.

C.2 Record the optical density of the unknown.

Substitute the values found in B.2 and C.2 and the concentration of the standard into Equation (1.5) which is derived from Beers law, to obtain the concentration of the sample.

$$\text{(1.5)} \qquad \begin{matrix}\text{concentration}\\ \text{of the sample}\end{matrix} = \left(\frac{\begin{matrix}\text{concentration}\\ \text{of the standard}\end{matrix}}{\begin{matrix}\text{optical density}\\ \text{of the standard}\end{matrix}} \right) \times \begin{matrix}\text{optical density}\\ \text{of the sample}\end{matrix}$$

To create a spectra for a colored liquid, sections A and B of the procedure above are repeated for a whole series of values of λ.

Exercise 1.1

We are given a vial of red liquid of unknown concentration and are told it contains reduced cytochrome c dissolved in water. We begin the analysis of this liquid by preparing a blank. After filling a cuvette with water and adjusting the spectrophotometer until the optical density read zero (steps A.1 and A.2), we prepare our standard by dissolving enough of the reduced cytochrome c in water to obtain a concentration of $1.99 \times 10^{-5} \frac{\text{moles}}{\text{liter}}$. We next insert the cuvette containing the standard into the spectrophotometer and record an optical density of 0.559 at a wavelength of 550 nm. Finally, we are ready to place the cuvette filled with the sample of reduced cytochrome c which has unknown concentration into the spectrophotometer (step C.1). An optical density of 0.662 at a wavelength of 550 nm is recorded (step C.2). Determine the concentration of the sample of reduced cytochrome c that was originally distributed.

Section 2 Concentration of Two Substances in a Colored Liquid

When using the procedure from Section 1 to determine the concentration of a substance, it was important to use a narrow band of wavelengths around the absorbing wavelength λ, because we were trying to closely approximate the conditions under which Beer's law holds and hence, when Equation (1.5) is valid. Also, when reading the optical density, the use of a narrow band of wavelengths around λ helps prevent the inclusion of other light absorbing substances which may be present in the solution. These considerations are especially important when two or more substances are present in the liquid. Figure 2.1 shows a set of hypothetical absorption curves for substances R and S in a solution at a given concentration.

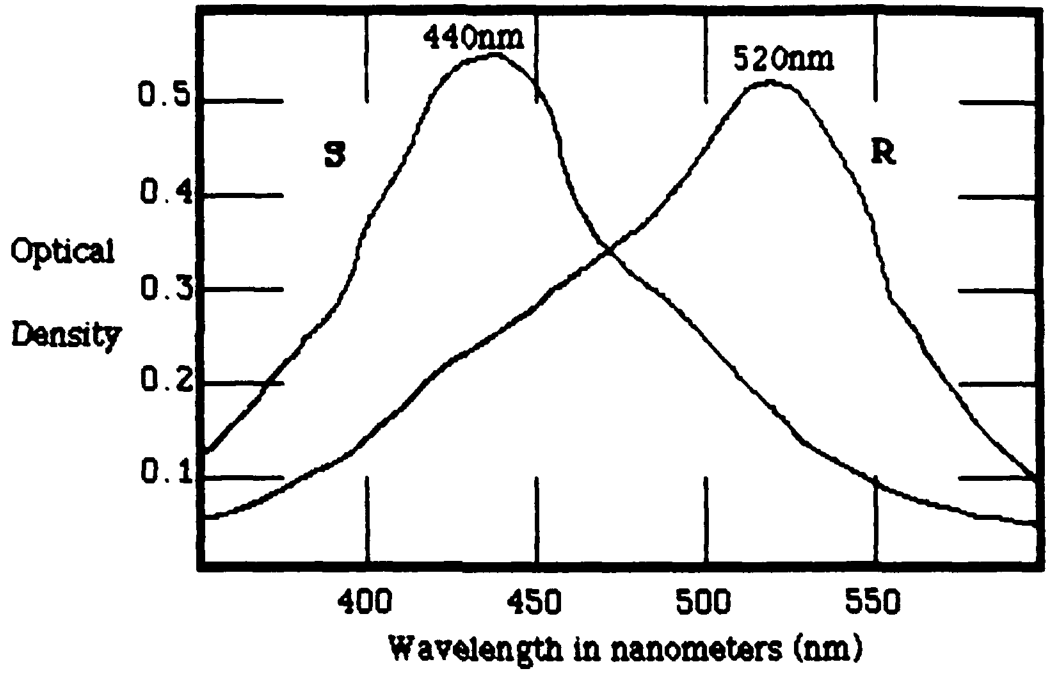


Figure 2.1

If the peak of each curve in Figure 2.1 did not overlap the tail of the other curve, then the optical density of each substance could be read directly from the graph by carefully selecting narrow bands around the peak wavelength. However, since the two curves do overlap (but do not coincide), we realize the optical density read off the spectrophotometer at 440 nm includes both the optical density of substance S and the optical density of substance R. Similarly, we can say the optical density reading taken from the spectrophotometer at 520 nm is made up of both the optical density of substance S at 520 nm and the optical density of substance R at 520 nm.

For colored liquids containing two substances, we will use the following technique to determine the concentration of each of the substances. This technique is valid when the absorption curves overlap, but do not coincide. We will denote the two substances as m and n, whose molar extinction coefficients are $\epsilon_{m\lambda_1}$ and $\epsilon_{n\lambda_1}$ at wavelength λ_1 and $\epsilon_{m\lambda_2}$ and $\epsilon_{n\lambda_2}$ at wavelength λ_2. Recall when we use ϵ_i, we are measuring the concentration in $\frac{\text{moles}}{\text{liter}}$ and the thickness of the solution (that is, the width of the cuvette) is 1 cm. If we apply Beer's law, we obtain

$$D_{m\lambda_1} = \epsilon_{m\lambda_1} c_m \quad \text{and} \quad D_{n\lambda_1} = \epsilon_{n\lambda_1} c_n \tag{2.1}$$

$$D_{m\lambda_2} = \epsilon_{m\lambda_2} c_m \quad \text{and} \quad D_{n\lambda_2} = \epsilon_{n\lambda_2} c_n \tag{2.2}$$

where $D_{m\lambda_1}$ and $D_{n\lambda_1}$ represent the light absorbed by the compounds m and n in the solution at wavelength λ_1. Here $D_{m\lambda_2}$ and $D_{n\lambda_2}$ represent the light absorbed by the compounds m and n in the solution at wavelength λ_2, and c_m and c_n are the concentrations of compound m and n respectively. Since there are two substances in the solution, they both contribute to the optical densities D_{λ_1} and D_{λ_2}, measured at the wavelengths λ_1 and λ_2, respectively. Thus, the equations for $D_{m\lambda_1}$ and $D_{n\lambda_1}$ are summed together, as are $D_{m\lambda_2}$ and $D_{n\lambda_2}$, producing Equations (2.3) and (2.4).

(2.3) $$D_{\lambda_1} = \epsilon_{m\lambda_1} c_m + \epsilon_{n\lambda_1} c_n$$

(2.4) $$D_{\lambda_2} = \epsilon_{m\lambda_2} c_m + \epsilon_{n\lambda_2} c_n$$

We observe that if the two curves do not overlap, then in Equation (2.3) either $\epsilon_{m\lambda_1}$ or $\epsilon_{n\lambda_1}$ is equal to zero at λ_1 and Equation (2.3) reduces to Equation (1.3), which is a form of the Lambert-Beer equation. Under similar conditions, Equation (2.4) will also reduce to such a form.

Now, we have two equation in two unknowns, c_m and c_n. We consider two methods which can be used to solve this system of equations and hence, determine the concentrations of the two substances.

Method 1

Solve Equation (2.3) for c_n

(2.5)
$$c_n = \frac{D_{\lambda_1} - \epsilon_{m\lambda_1} c_m}{\epsilon_{n\lambda_1}} .$$

Now substitute this into Equation (2.4).

(2.6)
$$D_{\lambda_2} = \epsilon_{m\lambda_2} c_m + \epsilon_{n\lambda_2} \left(\frac{D_{\lambda_1} - \epsilon_{m\lambda_1} c_m}{\epsilon_{n\lambda_1}} \right)$$

When we solve this equation for c_m, we obtain

(2.7)
$$c_m = \frac{\epsilon_{n\lambda_1} D_{\lambda_2} - \epsilon_{n\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_2} \epsilon_{n\lambda_1} - \epsilon_{m\lambda_1} \epsilon_{n\lambda_2}} .$$

Since all the terms on the right of Equation (2.7) are known quantities, we know c_m. Again, we observe that if the two curves do not overlap, then in the equation above, either $\epsilon_{n\lambda_1}$ or $\epsilon_{n\lambda_2}$ is equal to zero and the equation reduces to the form of the Lambert-Beer equation at λ_1 found in Section 1. To see this, suppose $\epsilon_{n\lambda_1} = 0$. Then, Equation (2.7) reduces to

(2.8) $$c_m = \frac{-\epsilon_{n\lambda_2} D_{\lambda_1}}{-\epsilon_{m\lambda_1} \epsilon_{n\lambda_2}} = \frac{D_{\lambda_1}}{\epsilon_{m\lambda_1}} ,$$

which can be simplified to obtain the desired equation.

(2.9) $$D_{\lambda_1} = \epsilon_{m\lambda_1} c_m$$

Exercise 2.1.

(a) Use a similar procedure to find the following expression for c_n.

(2.10) $$c_n = \frac{\epsilon_{m\lambda_1} D_{\lambda_2} - \epsilon_{m\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_1} \epsilon_{n\lambda_2} - \epsilon_{m\lambda_2} \epsilon_{n\lambda_1}}$$

(b) Show that if $\epsilon_{m\lambda_1} = 0$, then the equation for c_n reduces to the Lambert-Beer equation at λ_1.

(c) Show that if $\epsilon_{m\lambda_2} = 0$, then the equation for c_n reduces to the Lambert-Beer equation at λ_2.

Method 2

Equations (2.3) and (2.4) can be written as an augmented matrix.

$$(2.11)\qquad \left(\begin{array}{cc|c} \epsilon_{m\lambda_1} & \epsilon_{n\lambda_1} & D_{\lambda_1} \\ \epsilon_{m\lambda_2} & \epsilon_{n\lambda_2} & D_{\lambda_2} \end{array}\right)$$

This matrix can be row reduced to the following augmented matrix.

$$(2.12)\qquad \left(\begin{array}{cc|c} 1 & 0 & \dfrac{\epsilon_{n\lambda_1} D_{\lambda_2} - \epsilon_{n\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_2}\epsilon_{n\lambda_1} - \epsilon_{m\lambda_1}\epsilon_{n\lambda_2}} \\ 0 & 1 & \dfrac{\epsilon_{m\lambda_1} D_{\lambda_2} - \epsilon_{m\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_1}\epsilon_{n\lambda_2} - \epsilon_{m\lambda_2}\epsilon_{n\lambda_1}} \end{array}\right)$$

From this augmented matrix, we can determine the concentrations of m and n by using

$$(2.13)\qquad c_m = \frac{\epsilon_{n\lambda_1} D_{\lambda_2} - \epsilon_{n\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_2}\epsilon_{n\lambda_1} - \epsilon_{m\lambda_1}\epsilon_{n\lambda_2}}$$

and

(2.14) $$c_n = \frac{\epsilon_{m\lambda_1} D_{\lambda_2} - \epsilon_{m\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_1} \epsilon_{n\lambda_2} - \epsilon_{m\lambda_2} \epsilon_{n\lambda_1}} .$$

As expected, these formulas for c_m and c_n are exactly the same as the ones found by using Method 1 (Equation 2.7 and 2.10).

Exercise 2.2.

Verify the reduction of the matrix

(2.15) $$\left(\begin{array}{cc|c} \epsilon_{m\lambda_1} & \epsilon_{n\lambda_1} & D_{\lambda_1} \\ & & \\ \epsilon_{m\lambda_2} & \epsilon_{n\lambda_2} & D_{\lambda_2} \end{array}\right).$$

Exercise 2.3

This exercise is adapted from a laboratory study by Arnon (1949) where he determined the concentration of chlorophyll *a* and chlorophyll *b* which were both dissolved in the same liquid. The two absorption peaks, where the spectrophotometer was set during this experiment, are:

Chlorophyll *a* - at wavelength 663 nm exhibits an extinction coefficient of 82.04 $\frac{\text{liters}}{\text{mole}}$ and at wavelength 645 nm exhibits an extinction coefficient of 16.45 $\frac{\text{liters}}{\text{mole}}$ and

Chlorophyll *b*- at wavelength 663 nm exhibits an extinction coefficient of 9.27 $\frac{\text{liters}}{\text{mole}}$ and at wavelength 645 nm exhibits an extinction coefficient of 45.6 $\frac{\text{liters}}{\text{mole}}$.

The optical density measured at wavelength 663 nm is 0.506 and the optical density measured at 645 nm is 0.187. Using this information, determine the concentration of chlorophyll *a* and chlorophyll *b* dissolved in the same liquid. (Be sure to indicate the value of each variable.)

Exercise 2.4

Suppose we have three substances a, b, and p, dissolved in a single solution and their spectra overlap, but none coincide. By using a similar procedure to the one used to find the optical density of a liquid containing two substances, find the three equations for optical density (they are similar to Equations (2.3) and (2.4)). Use the following variables to create these equations.

Let: $\lambda_1, \lambda_2, \lambda_3$ represent the absorbing wavelength of each of the three substances,

$\epsilon_{a\lambda_1}, \epsilon_{b\lambda_1}, \epsilon_{p\lambda_1}$ represent the extinction coefficients for each substance at wavelength λ_1,

$\epsilon_{a\lambda_2}, \epsilon_{b\lambda_2}, \epsilon_{p\lambda_2}$ represent the extinction coefficients for each substance at wavelength λ_2,

$\epsilon_{a\lambda_3}, \epsilon_{b\lambda_3}, \epsilon_{p\lambda_3}$ represent the extinction coefficients for each

substance at wavelength λ_3,

c_a, c_b, c_p represent the concentrations of the three substances, and

$D_{\lambda_1}, D_{\lambda_2}, D_{\lambda_3}$ represent the optical densities measured at each

wavelength.

The system of three equations found in Exercise 2.4 can be written as an *augmented matrix and reduced.* *From this reduced augmented matrix we* can obtain the formulas needed to determine the individual concentrations of the three substances. This task is not hard but becomes very cumbersome because there are so many variables involved. The augmented matrix is reduced until the coefficient matrix is in upper triangular form.

$$(2.16)\quad \left(\begin{array}{ccc|c} 1 & \dfrac{\epsilon_{b\lambda_1}}{\epsilon_{a\lambda_1}} & \dfrac{\epsilon_{p\lambda_1}}{\epsilon_{a\lambda_1}} & \dfrac{D_{\lambda_1}}{\epsilon_{a\lambda_1}} \\ 0 & 1 & \dfrac{\epsilon_{a\lambda_1}\epsilon_{p\lambda_2} - \epsilon_{a\lambda_2}\epsilon_{p\lambda_1}}{\epsilon_{a\lambda_1}\epsilon_{b\lambda_2} - \epsilon_{a\lambda_2}\epsilon_{b\lambda_1}} & \dfrac{\epsilon_{a\lambda_1}D_{\lambda_2} - \epsilon_{a\lambda_2}D_{\lambda_1}}{\epsilon_{a\lambda_1}\epsilon_{b\lambda_2} - \epsilon_{a\lambda_2}\epsilon_{b\lambda_1}} \\ 0 & 0 & 1 & \Re \end{array}\right)$$

where $\Re = \dfrac{A+B}{C+D}$ and $A = \left(\epsilon_{a\lambda_1} D_{\lambda_3} - \epsilon_{a\lambda_3} D_{\lambda_1}\right)\left(\epsilon_{a\lambda_1} \epsilon_{b\lambda_2} - \epsilon_{a\lambda_2} \epsilon_{b\lambda_1}\right)$

$$B = \left(\epsilon_{a\lambda_1} D_{\lambda_2} - \epsilon_{a\lambda_2} D_{\lambda_1}\right)\left(\epsilon_{a\lambda_1} \epsilon_{b\lambda_3} - \epsilon_{a\lambda_3} \epsilon_{b\lambda_1}\right)$$

$$C = \left(\epsilon_{a\lambda_1} \epsilon_{p\lambda_3} - \epsilon_{a\lambda_3} \epsilon_{p\lambda_1}\right)\left(\epsilon_{a\lambda_1} \epsilon_{b\lambda_2} - \epsilon_{a\lambda_2} \epsilon_{b\lambda_1}\right)$$

$$D = \left(\epsilon_{a\lambda_1} \epsilon_{p\lambda_2} - \epsilon_{a\lambda_2} \epsilon_{p\lambda_1}\right)\left(\epsilon_{a\lambda_1} \epsilon_{b\lambda_3} - \epsilon_{a\lambda_3} \epsilon_{b\lambda_1}\right)$$

Exercise 2.5

A vial of green liquid is brought in for analysis which contains three substances, chlorophyll *a* (denoted by a), chlorophyll *b* (denoted by b) and protochlorophyll (denoted by p) dissolved in it. Optical density readings from the spectrophotometer are taken at the following wavelengths: $\lambda_1 = 660\,\text{nm}$, $\lambda_2 = 640\,\text{nm}$, and $\lambda_3 = 620\,\text{nm}$. During analysis the optical densities are found to be: $D_{\lambda_1} = 0.21$, $D_{\lambda_2} = 0.72$, and $D_{\lambda_3} = 0.53$.

Extinction coefficients for the three substances at their respective wavelengths are:

wavelength (nm)	a $(\frac{\text{liters}}{\text{mole}})$	b $(\frac{\text{liters}}{\text{mole}})$	p $(\frac{\text{liters}}{\text{mole}})$
660	96	5	1
640	15	58	1
620	14	10	40

Determine the concentration of each substance in the vial. Hint: first find c_p and then use back substitution and the information in the augmented matrix in Equation (2.16).

Exercise 2.6

Discuss the conditions under which the technique to find the concentration of three substances dissolved in one liquid can be generalized to find the concentration of n substances dissolved in a single liquid.

Exercise 2.7

Discuss some of the advantages in using matrices to find the concentration of substances dissolved in a liquid.

References

Arnon, D. I. (1949). Copper enzymes in isolated chloroplasts, polyphenoloxidase in *Beta vulgaris*. Plant Physiology, 24(1), 1-15.

Snell, F. D., & Snell, C. T. (1948). Colorimetric methods of analysis (Vol I) (3rd ed.). Princeton, NJ: D. Van Nostrand.

Application I Appendix: Solutions to Exercises

Exercise 1.1

$$\text{concentration of reduced cytochrome c} = \frac{1.99 \times 10^{-5}}{0.559} \times 0.662 = 2.36 \times 10^{-5} \frac{\text{moles}}{\text{liter}}$$

Exercise 2.1

(a) Solve Equation (2.4) for c_m

$$c_m = \frac{D_{\lambda_2} - \epsilon_{n\lambda_2} c_n}{\epsilon_{m\lambda_2}} \qquad \text{(A.1)}$$

Now substitute this into Equation (2.3).

$$D_{\lambda_1} = \epsilon_{m\lambda_1} \left(\frac{D_{\lambda_2} - \epsilon_{n\lambda_2} c_n}{\epsilon_{m\lambda_2}} \right) + \epsilon_{n\lambda_1} c_n \qquad \text{(A.2)}$$

When we solve this equation for c_n, we obtain

$$c_n = \frac{\epsilon_{m\lambda_1} D_{\lambda_2} - \epsilon_{m\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_1} \epsilon_{n\lambda_2} - \epsilon_{m\lambda_2} \epsilon_{n\lambda_1}} \qquad \text{(A.3)}$$

(b) Letting $\epsilon_{m\lambda_1} = 0$, the equation for c_n reduces to

(A.4)
$$c_n = \frac{-\epsilon_{m\lambda_2} D_{\lambda_1}}{-\epsilon_{m\lambda_2} \epsilon_{n\lambda_1}} = \frac{D_{\lambda_1}}{\epsilon_{n\lambda_1}} \quad ,$$

which can be simplified to obtain the desired equation.

(A.5)
$$D_{\lambda_1} = \epsilon_{n\lambda_1} c_n$$

(c) Letting $\epsilon_{m\lambda_2} \to 0$, the equation for c_n reduces to

(A.6)
$$c_n = \frac{\epsilon_{m\lambda_1} D_{\lambda_2}}{\epsilon_{m\lambda_1} \epsilon_{n\lambda_2}} = \frac{D_{\lambda_2}}{\epsilon_{n\lambda_2}} \quad ,$$

which can be simplified to obtain the desired equation.

(A.7)
$$D_{\lambda_2} = \epsilon_{n\lambda_2} c_n$$

<u>Exercise 2.2</u>

Multiply the first row by $\dfrac{1}{\epsilon_{m\lambda_1}}$.

$$(A.8) \quad \left(\begin{array}{cc|c} 1 & \frac{\epsilon_{n\lambda_1}}{\epsilon_{m\lambda_1}} & \frac{D_{\lambda_1}}{\epsilon_{m\lambda_1}} \\ \epsilon_{m\lambda_2} & \epsilon_{n\lambda_2} & D_{\lambda_2} \end{array} \right)$$

Multiply row one by $-\epsilon_{m\lambda_2}$ and add it to row two.

$$(A.9) \quad \left(\begin{array}{cc|c} 1 & \frac{\epsilon_{n\lambda_1}}{\epsilon_{m\lambda_1}} & \frac{D_{\lambda_1}}{\epsilon_{m\lambda_1}} \\ 0 & \frac{\epsilon_{m\lambda_1}\epsilon_{n\lambda_2} - \epsilon_{m\lambda_2}\epsilon_{n\lambda_1}}{\epsilon_{m\lambda_1}} & \frac{\epsilon_{m\lambda_1}D_{\lambda_2} - \epsilon_{m\lambda_2}D_{\lambda_1}}{\epsilon_{m\lambda_1}} \end{array} \right)$$

Multiply row two by $\dfrac{\epsilon_{m\lambda_1}}{\epsilon_{m\lambda_1}\epsilon_{n\lambda_2} - \epsilon_{m\lambda_2}\epsilon_{n\lambda_1}}$ to obtain

$$\text{(A.10)} \qquad \left(\begin{array}{cc|c} 1 & \dfrac{\epsilon_{n\lambda_1}}{\epsilon_{m\lambda_1}} & \dfrac{D_{\lambda_1}}{\epsilon_{m\lambda_1}} \\ & & \\ 0 & 1 & \dfrac{\epsilon_{m\lambda_1} D_{\lambda_2} - \epsilon_{m\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_1}\epsilon_{n\lambda_2} - \epsilon_{m\lambda_2}\epsilon_{n\lambda_1}} \end{array}\right).$$

Multiply row two by $-\dfrac{\epsilon_{n\lambda_1}}{\epsilon_{m\lambda_1}}$ and add it to row one.

$$\text{(A.11)} \qquad \left(\begin{array}{cc|c} 1 & 0 & \dfrac{\epsilon_{n\lambda_1} D_{\lambda_2} - \epsilon_{n\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_2}\epsilon_{n\lambda_1} - \epsilon_{m\lambda_1}\epsilon_{n\lambda_2}} \\ & & \\ 0 & 1 & \dfrac{\epsilon_{m\lambda_1} D_{\lambda_2} - \epsilon_{m\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_1}\epsilon_{n\lambda_2} - \epsilon_{m\lambda_2}\epsilon_{n\lambda_1}} \end{array}\right)$$

This is Equation (2.12).

<u>Exercise 2.3</u>

We define the following variables which we will use in Equations (2.13) and (2.14).

Let m = chlorophyll *a* $\qquad$ n = chlorophyll *b*

$\lambda_1 = 663$ nm $\qquad$ $\lambda_2 = 645$ nm

$\epsilon_{m\lambda_1} = 82.04 \dfrac{\text{liters}}{\text{mole}}$ $\qquad$ $\epsilon_{m\lambda_2} = 16.45 \dfrac{\text{liters}}{\text{mole}}$

$\epsilon_{n\lambda_1} = 9.27 \dfrac{\text{liters}}{\text{mole}}$ $\qquad$ $\epsilon_{n\lambda_2} = 45.6 \dfrac{\text{liters}}{\text{mole}}$

$D_{\lambda_1} = 0.506$ $\qquad$ $D_{\lambda_2} = 0.187$

$$c_m = \frac{\epsilon_{n\lambda_1} D_{\lambda_2} - \epsilon_{n\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_2}\epsilon_{n\lambda_1} - \epsilon_{m\lambda_1}\epsilon_{n\lambda_2}} \tag{A.12}$$

$$= \frac{(9.27)(0.187) - (45.6)(0.506)}{(16.45)(9.27) - (82.04)(45.6)} \; \frac{-21.340}{-3588.533}$$

$$= 5.95 \times 10^{-3} \frac{\text{moles}}{\text{liter}}$$

$$c_n = \frac{\epsilon_{m\lambda_1} D_{\lambda_2} - \epsilon_{m\lambda_2} D_{\lambda_1}}{\epsilon_{m\lambda_1}\epsilon_{n\lambda_2} - \epsilon_{m\lambda_2}\epsilon_{n\lambda_1}} \tag{A.13}$$

$$= \frac{(82.04)(0.187) - (16.45)(0.506)}{(82.04)(45.6) - (16.45)(9.27)} = \frac{7.018}{3588.533}$$

$$= 1.96 \times 10^{-3} \frac{\text{moles}}{\text{liter}}$$

Exercise 2.4

(A.14) $$D_{\lambda_1} = \epsilon_{a\lambda_1} c_a + \epsilon_{b\lambda_1} c_b + \epsilon_{p\lambda_1} c_p$$

(A.15) $$D_{\lambda_2} = \epsilon_{a\lambda_2} c_a + \epsilon_{b\lambda_2} c_b + \epsilon_{p\lambda_2} c_p$$

(A.16) $$D_{\lambda_3} = \epsilon_{a\lambda_3} c_a + \epsilon_{b\lambda_3} c_b + \epsilon_{p\lambda_3} c_p$$

Exercise 2.5

$$c_p = \frac{A+B}{C+D} = \frac{263334.42 + 58713.3}{21016218 + 72090} = \frac{322047.72}{21088308} = 1.5 \times 10^{-2} \frac{\text{moles}}{\text{liter}}$$

$$c_b = \frac{65.97 - 81(.015)}{5493} = \frac{64.755}{5493} = 1.2 \times 10^{-2} \frac{\text{moles}}{\text{liter}}$$

$$c_a = \frac{.21 - 5(.012) - 1(.015)}{96} = \frac{.135}{96} = 1.4 \times 10^{-3} \frac{\text{moles}}{\text{liter}}$$

Exercise 2.6

The main condition under which the technique to find the concentration of substances dissolved in a liquid can be generalized, is that the absorption curves of the n substances do not coincide. However, some or all may overlap. Theoretically then, it appears that we could generalize this technique. From the practical aspect, the value of n would be determined by the accuracy of the spectrophotometer. Also, the more substances present the harder it will become to determine the equations to find the concentration of each substance in the solution. (This is not a factor if a computer is used. See the solution to Exercise 2.7)

Exercise 2.7

When we tried to reduce the augmented matrix for n=3, we found the task very cumbersome. It was mainly because we were trying to find a formula so the concentration could be obtained by simply substituting in the known values. However, if a computer is used to reduce the system, a general formula is not needed because the computer can reduce a matrix as easily as it can substitute values into a formula and calculate the result. Thus, the values for each variable can be substituted directly into the augmented matrix and the system reduced, from which the values of the concentrations can be determined.

APPENDIX B

Application II

Contents of Application II

Application II

Linear Algebra Applied to Graph Theory

A New Look at Solving the Transportation Problem

Linear Algebra Prerequisites: The knowledge of definitions and theorems involving linear independence, linear dependence, spanning, and basis.

Graph Theory Prerequisites: None.

Section 1 Introduction

The famous Mr. Potatohead Potato Chip Company of Mountain Home, Idaho, recently developed a new line of crunchy style chips called West Coast Krunchers. To promote their new chips in the western United States, Mr. Potatohead is negotiating with SuperFoods, a big food store chain, to sell their chips. Mr. Potatohead has two warehouses: one in Boise, Idaho, which has 1900 cases filled with bags of West Coast Krunchers ready to ship, and one in Modesto, California which has 1200 cases of chips. SuperFoods has 3 central warehouses which supply all their local stores; they are located in Colorado Springs, Colorado, and in San Francisco and San Diego, California. The Boise warehouse can ship West Coast Krunchers to Colorado Springs for $.17 a case, to San Francisco for $.18 a case, and to San Diego for $.23 a case. The warehouse in Modesto can ship Krunchers to Colorado Springs for $.25 a case, to San Francisco for $.15 a case, and to San Diego for $.21 a case. The SuperFood warehouse in Colorado Springs has ordered 1000 cases, while San Francisco needs 1200 cases, and San Diego only wants 900 cases. To determine how much profit SuperFoods can make on the West Coast Krunchers, the buying agent needs to know the least expensive way to get the chips from the two Mr. Potatohead warehouses to their own warehouses.

This shipping problem is an example of a transportation problem; the shipping routes in this problem can be represented by the diagram in Figure 1.1. We will be able to solve transportation problems using the techniques from Graph Theory, a subject which we are about to study.

Graph Theory is an area of mathematics which deals with properties of structures such as the one illustrated in Figure 1.1.

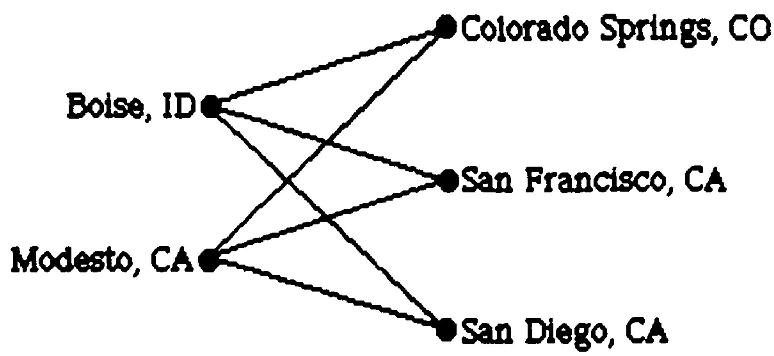


Figure 1.1

Before beginning to explore a new subject such as Graph Theory, it is important to understand and be able to use the fundamental definitions which comprise Graph Theory. The following definitions will provide us with the language necessary to discuss problems such as the transportation problem above.

A **graph** G consists of a finite nonempty set V, whose members are called **vertices**, together with a finite (possibly empty) set E of **edges**, where each edge in the set E consists of a two-element subset of V. We often depict a graph by a diagram in which a dot represents a vertex and a line connecting two dots represents the edge consisting of the subset which contains the two corresponding vertices. In the graph G in Figure 1.2, the set of vertices is $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the set of edges is

$E = \{ \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\} \}$. As you can see, the notation for edges can be quite cumbersome at times, so that often we use notation such as e_1 to represent the edge $\{v_1, v_2\}$ and we say that the edge e_1 **joins** the vertices v_1 and v_2. Using this shorthand notation, the set of edges E in Figure 1.2 can be written as $E = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7 \}$.

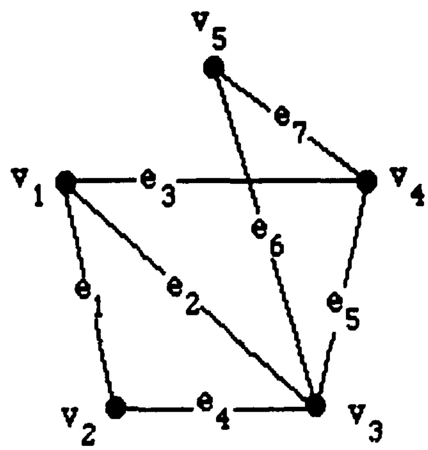


Figure 1.2

If edge e_i joins two vertices v_m and v_n, then v_m and e_i are said to be **incident**, as are v_n and e_i. Furthermore, the vertices v_m and v_n are said to be **adjacent** vertices. In Figure 1.2, the three edges which are incident to v_1 are e_1, e_2 and e_3. If v_m is a vertex of a graph G, then the **degree** of v_m is the number of edges that are incident to v_m. For example, in Figure 1.2, the degree of v_1 is 3, while the degree of v_2 is 2.

The following concepts will help us understand the structure of graphs. A **walk** from vertex v_1 to vertex v_k in a graph G is an alternating sequence of vertices and edges of the form

$$v_1, e_1, v_2, e_2, \dots, v_{k-1}, e_{k-1}, v_k$$

where each e_i is incident to v_i and v_{i+1} and $k \geq 1$. We sometimes denote a walk by listing only the sequence of vertices; for example, the walk above would be written as $v_1, v_2, \dots, v_{k-1}, v_k$. A **path** from v_1 to v_k in a graph G is a walk in which no vertex or edge is repeated. Thus, one path from v_1 to v_3 in Figure 1.2 would be v_1, v_4, v_3. A **cycle** is a walk $v_1, v_2, \dots, v_{k-1}, v_k$ where $k \geq 3$ in which all vertices and edges are distinct except that $v_1 = v_k$. For example, v_1, v_2, v_3, v_1 is a cycle in Figure 1.2. A graph G is **connected** if there is a walk between any two vertices, otherwise G is said to be **disconnected**. Therefore, we see that the graph G in Figure 1.2 is connected. Also, any graph in which every pair of distinct vertices is joined by an edge is called **complete**. Notice that the graph in Figure 1.2 is not complete, since there is no edge joining the pairs of vertices v_1 and v_5, v_2 and v_4, or v_2 and v_5.

Exercise 1.1

Use Figure 1.2 to find each of the following.

(a) What is the degree of v_3? v_4? v_5?

(b) Find a walk which is not a path from v_2 to v_5.

(c) Create a path from the walk in (b) above.

(d) Find a cycle with 3 distinct vertices and one with 4 vertices.

Exercise 1.2

(a) Draw a connected graph.

(b) Draw a disconnected graph.

(c) Draw a complete graph with 4 vertices.

(d) Is it possible to draw a complete disconnected graph?

Exercise 1.3

Prove that if G is a connected graph, then there is a path between any two distinct vertices of G.

Sometimes we are interested in a graph which is contained within a larger graph. A graph H is a **subgraph** of a graph G if the vertex and edge sets of H are contained in the vertex and edge sets of G. By examining the graph in Figure 1.2, we see that one example of a subgraph of G would consist of the set of vertices { v_1, v_2, v_3, v_4 } and the set of edges

{ e_1, e_2, e_3, e_4 }. If a subgraph H of a graph G has the same set of vertices as G, then H is called a **spanning subgraph** of G. If we consider the graph in Figure 1.2, an example of a spanning subgraph would consist of the set of vertices { v_1, v_2, v_3, v_4, v_5 } and the set of edges { e_3, e_4, e_5, e_6, e_7 }.

Exercise 1.4

(a) Draw a subgraph of the graph in Figure 1.2 which is not a spanning subgraph.

(b) Draw a spanning subgraph of the graph in Figure 1.2.

Section 2 Trees and Counting the Number of Spanning Trees in a Graph

A graph which is connected and has no cycles is called a **tree**. Trees are the simplest, yet most important, class of connected graphs. A spanning subgraph of a graph G which is also a tree is called a **spanning tree** of G. The two graphs in Figure 2.1 are spanning trees of the graph G of Figure 1.2. We will spend much of the remainder of this study investigating the idea of spanning trees and see how they can be used to solve transportation problems like the Mr. Potatohead potato chip problem of Section 1.

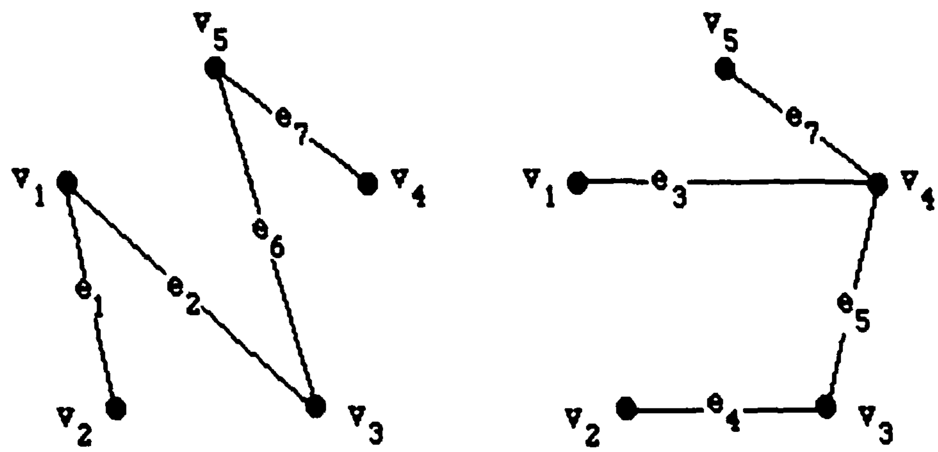


Figure 2.1

Exercise 2.1

(a) Using the graph in Figure 1.2, draw a tree which is not a spanning tree.

(b) Are all trees necessarily spanning trees?

(c) Are all spanning trees necessarily trees?

Exercise 2.2

For the graph G in Figure 2.2 find as many spanning trees for G as you can.

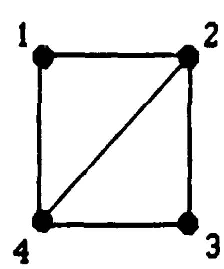


Figure 2.2

From the above exercise we see that employing the definition to find all the spanning trees for a graph raises an important question. How do we know when we have found all the spanning trees of a graph? We can answer this question by looking at an application of *determinants*. The concept of determinants is introduced in linear algebra and involves the use of matrices. The following algorithm will help us construct the matrix which is used in this surprisingly simple technique to count the number of spanning trees in a graph. A proof that this algorithm counts the number of spanning trees can be found in Harary (1969, pp.152-153).

Algorithm To Find the Number of Spanning Trees in a Graph G:

1. Label the vertices of G as 1, 2, ... , n+1.
2. Construct an $(n+1)\times(n+1)$ matrix A as follows:

   (i) for the diagonal entry A_{ii}, for i= 1,2,...,n+1, let

   A_{ii} be the degree of vertex i in G.

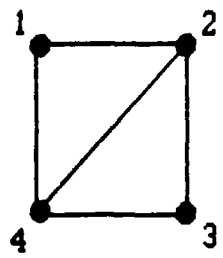
   (ii) for each off-diagonal entry A_{ij}, let

$$A_{ij} = \begin{cases} 0 & \text{if vertices } i \text{ and } j \text{ are not adjacent in } G \\ -1 & \text{if vertices } i \text{ and } j \text{ are adjacent in } G \end{cases}$$

3. Let D be the n x n matrix obtained by deleting row n+1 and column n+1 from A.
4. The number of spanning trees of G is given by $|D|$, the determinant of D.

Example

Recall the graph G in Figure 2.2. Use the four steps in the algorithm above to count the number of spanning trees in the graph G.

1. 

2. $$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

3. $$D = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

4. The number of spanning trees of G is $|D| = 8$.

Compare the eight spanning trees given in Figure 2.3 with the spanning trees that you found in Exercise 2.1. Were you able to find all the spanning trees?

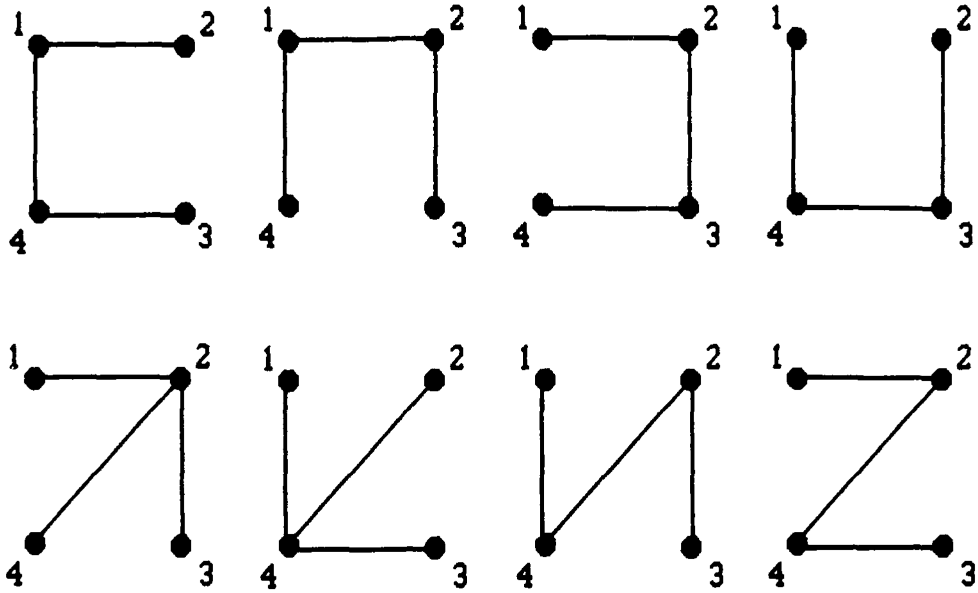


Figure 2.3

Exercise 2.3

Using the determinant method described above, find the number of spanning trees for the graph G, in Figure 2.4. Draw all the spanning trees of G.

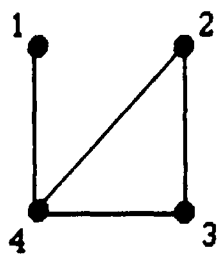


Figure 2.4

Exercise 2.4

Find the number of spanning trees for the graph G, in Figure 2.5. Draw all the spanning trees of G.

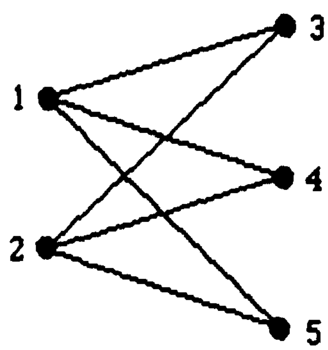


Figure 2.5

As a point of interest, the graph in Figure 2.5 is called a *complete bipartite graph* and is the type of graph which was used to represent the Mr. Potatohead potato chip transportation problem in Section 1. In general, a graph G is a **bipartite graph** if the set of vertices V can be grouped into two sets, V_1 and V_2, such that $V = V_1 \cup V_2$ and where

1) V_1 and V_2 are not empty,

2) $V_1 \cap V_2 = \emptyset$, and

3) each edge in G joins a vertex of V_1 with a vertex of V_2.

Such a bipartite graph is a **complete bipartite graph** if each vertex in V_1 is adjacent to each vertex in V_2.

Example

To show that the graph in Figure 2.5 is a complete bipartite graph, we must first show that it is a bipartite graph. If we let the set V_1 consist of the vertices 1 and 2 and let V_2 be the set of vertices 3, 4, and 5, then V_1 and V_2 satisfy the definition of a bipartite graph. We also see that each vertex in V_1 is adjacent to each vertex in V_2. Hence the graph in Figure 2.5 is a complete bipartite graph.

In a later section we will study techniques which will allow us to solve transportation problems by looking at the spanning trees of complete bipartite graphs.

Example

Figure 2.6 shows a bipartite graph which is not a complete bipartite graph.

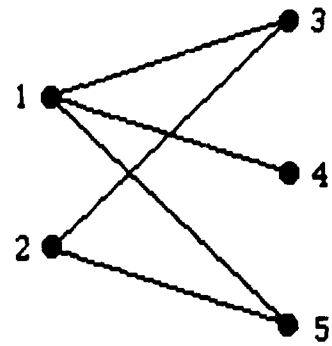


Figure 2.6

Section 3 Properties of Spanning Trees

We will now establish some basic properties of spanning trees by proving a series of four theorems and two accompanying lemmas. It is important to keep in mind the statement of these theorems, as we will be using them later to determine some very interesting relationships. The proofs of these theorems have been expressly written in a non-rigorous tone to enhance their readability.

Lemma 1

Let G be a connected graph. If an edge e of G lies on a cycle of G, then G - e (which denotes the graph G without the edge e) is connected.

Proof

Let G be a connected graph with at least one cycle. Let e = (v_1, v_2) be an edge on a cycle of G, and let that cycle consist of the edge e and a path P between v_1 and v_2. In order to show that G - e is connected we must show that there is a walk in G - e between any two vertices. Choose any two vertices u and v. Because G is connected, we know there exists a walk in G from u to v. If e is not an edge on this walk, then the walk from u to v is still a walk in the graph G - e. If e is an edge on the walk, then replacing each appearance of e, in the walk, by the path P, yields a new walk from u to v that does not contain e. In either case, we have found a walk in G - e from u to v. Since the two vertices u and v were chosen arbitrarily, the graph G - e is connected. □

Theorem 1

Every connected graph G contains a spanning tree.

Proof

Let G be any connected graph. If G has any cycles, then by Lemma 1, removing any edge in a cycle yields a connected graph with fewer edges. Repeating this procedure eventually yields a connected subgraph G* of G that has no cycles. Thus, G* is a spanning tree of G. □

Lemma 2

In any tree S with two or more vertices, there is at least one vertex of degree 1.

Proof

We will prove this theorem using the technique of proof by contradiction. Let S be a tree with two or more vertices. Because S is connected, no vertex has degree 0, so every vertex has degree at least one. Suppose that no vertex of S has degree one, that is, the degree of every vertex of S is greater than one. Choose any vertex v in S, and construct a walk starting from v as follows. Since the degree of v is at least 2, we may choose one of the edges incident with v, call it e_1, and proceed to a new vertex v_1.

Because the degree of v_1 is also at least 2, there is at least one other edge

$e_2 \neq e_1$ incident with v_1 which we may choose to continue the walk.

However, if we continue on in this manner, since the number of vertices is finite we must eventually return to some vertex which was previously made part of the walk, thus creating a cycle. But this contradicts the assumption that S is a tree. Therefore there must be at least one vertex of degree 1. □

Theorem 2

Let S be a tree with n+1 vertices. Then S has n edges.

Proof

We will prove this theorem by induction:

(1) Show that the statement is true for n=0.

(2) Assume the statement is true for n=k (induction hypothesis).

(3) Show that the statement is true for n=k+1.

To show (1), let S have 1 vertex, then S has n-0 edges. Assume the statement is true for n-k and show (3). Let S be a tree with k+1 vertices. We must show that S has k edges. By Lemma 2, there is a vertex v with degree 1. Then the graph obtained by deleting the vertex v and the incident edge from S is a tree with k vertices and by the induction hypothesis, must have k-1 edges. Since this graph was obtained by deleting one vertex and edge from S, S must have k edges. □

Theorem 3

Any pair of vertices in a tree is joined by exactly one path.

Proof

Let S be any tree. Then S is connected and so by Exercise 1.1, there is a path between any two vertices. To show that there is exactly one path, we will use the method of proof by contradiction. Let u and v be two vertices in S, and suppose there are two distinct paths, P_1 and P_2, from u to v.

Because $P_1 \neq P_2$, there must be a vertex w_1 (possibly $w_1 - u$) lying in both P_1 and P_2, such that the next vertex in P_1 is not in P_2. That is, the two paths separate at the vertex w_1. We continue along the path P_1 until we reach the first vertex w_2 (possibly $w_2 - v$) which is on both paths. Now consider the part of the path P_1 between w_1 and w_2 and the part of the path P_2 between w_1 and w_2. These parts form a cycle. But this contradicts the fact that S is a tree and hence has no cycles. Therefore any two vertices in a tree are joined by exactly one path. □

Theorem 4

Let G be a connected graph with n+1 vertices and S be a spanning subgraph of G. If S has two of the following properties, then it also has the third, thus making S a spanning tree.

(a) S has n edges

(b) S is connected

(c) S has no cycles

Proof

To prove this theorem we must assume any two of the properties are true and prove that the third property is true. Thus, we have three parts to prove:

Part 1. Assume (a) and (b), show (c).

To do this we will use the method of proof by contradiction. Let S be a connected spanning subgraph with n+1 vertices and n edges. Suppose S has one or more cycles. Because S is connected, by Theorem 1, we know we can find a spanning tree T of S. In doing so, we will delete at least one edge, so T must have less than n edges. But this contradicts Theorem 2; therefore, S has no cycles.

Part 2. Assume (b) and (c), show (a).

Let S be a connected spanning subgraph with n+1 vertices and no cycles. By definition, S is a spanning tree. So by Theorem 2, S has n edges.

Part 3. Assume (a) and (c), show (b).

Let S be a spanning subgraph with n+1 vertices, n edges and no cycles. Show that S is connected. We will use the method of proof by contradiction. Suppose S is not connected. Then S can be viewed as a series of connected subgraphs of S denoted by $S_1, S_2, \ldots, S_k$ where $k > 1$.

Let $n_1, n_2, \ldots, n_k$, represent the number of vertices in $S_1, S_2, \ldots, S_k$, respectively. Then $\sum_{i=1}^{k} n_i = n+1$. Since S has no cycles, each S_i for $1 \leq i \leq k$ has no cycles. By Theorem 2, $S_1, S_2, \ldots, S_k$ have $n_1 - 1, n_2 - 1, \ldots, n_k - 1$ edges, respectively. Thus, the number of edges in S is

$$\sum_{i=1}^{k} (\text{the number of edges in } S_i) = \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = (n+1)-k$$

where $k > 1$. However, this contradicts the assumption that S has n edges. Therefore, S is connected. □

Section 4 Relationships Between Graph Theory and Linear Algebra

Many concepts in Linear Algebra are very general in nature. When studying other areas of mathematics, such as Graph Theory, we sometimes come across concepts which are very similar to concepts in Linear Algebra. If we can take a general concept in Linear Algebra and tailor it to fit a situation in Graph Theory, then we can gain new insights into the concepts. Thus, we will be able to get a richer understanding of Graph Theory, by viewing properties of graphs in terms of Linear Algebra. To better see this, recall the definition of a spanning tree and the statements of Theorems 2-4; we will compare these to their counterparts in Linear Algebra.

<u>**Linear Algebra**</u>	<u>**Graph Theory**</u>
Definition A basis for a vector space V is a set of vectors which 1. are linearly independent and 2. span the vector space.	**Definition** A spanning tree of a graph G is a spanning subgraph of G which 1. has no cycles and 2. is connected.
Theorem Let V be a vector space with dimension n. Then every basis of V has n elements.	**Theorem 2** Let G be a graph with n+1 vertices. Then every spanning tree of G has n edges.
Theorem Each vector in a vector space is a unique linear combination of vectors in any basis.	**Theorem 3** Every two vertices in a graph are joined by exactly one path in any spanning tree.

Theorem

Let V be a vector space of dimension n and S be a set of vectors in V. If S has any two of the following properties, then it also has the third, and thus forms a basis for V.

(a) S contains n vectors.

(b) S spans V.

(c) The vectors in S are linearly independent.

Theorem 4

Let G be a connected graph with n+1 vertices and S be a spanning subgraph of G. If S has two of the following properties, then it also has the third, and thus is a spanning tree.

(a) S has n edges.

(b) S is connected.

(c) S has no cycles.

It is not a coincidence, of course, that we can make this comparison. We have been working with a new form of a vector space, where:

an edge represents a vector,

a spanning tree represents a basis,

a connected spanning subgraph represents a spanning set and

a subgraph with no cycles represents a linearly independent set.

(At this point, it may be helpful to review the previous definition and theorems from Graph Theory and their Linear Algebra counterparts.)

To get an indication of how this works (a full explanation is beyond the scope of this study), we must first develop notation to describe a vector in this vector space. We will only be interested in vector spaces which correspond to complete graphs. We will use the same symbol G to denote both the vector space and the corresponding graph.

Select a complete graph G with n+1 vertices, and label the vertices $v_1, v_2, \ldots, v_{n+1}$. A vector in the "vector space" G which corresponds to the complete graph G will be described by an (n+1)-tuple all of whose entries are zero or one, with there being an even number of ones in each tuple. To visualize the relationship between the vectors in the vector space G and the edges of the graph G consider the complete graph in Figure 4.1.

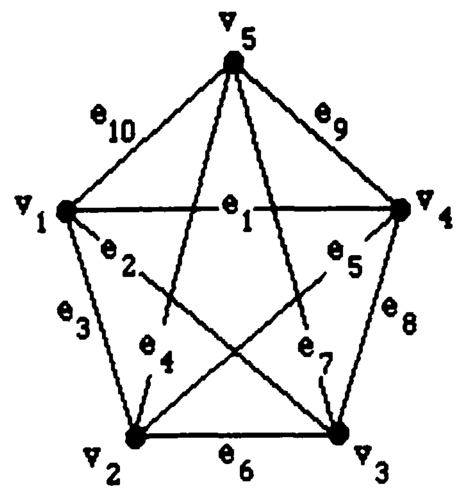


Figure 4.1

Since there are five vertices in the graph, we know that n=4, thus the vectors are 5-tuples. An edge joining vertices i and j in the graph, is represented by the vector which has a '1' for its i^{th} and j^{th} entries, and '0' elsewhere. To remind us that a vector in the vector space corresponds to a certain edge in the graph, we will use the same notation to denote the vector which we use for the edge in the graph. For example, the edge e_1 which joins the vertices the v_1 and v_4 in the graph corresponds to the

5-tuple $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, while $e_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ represents the edge, which joins the vertices v_2 and v_4.

In any vector space, it is necessary to be able to add vectors together. How can we interpret the addition of two vectors (which correspond to two edges) in a vector space (which corresponds to a graph)? If we refer back to Theorem 3 and its Linear Algebra counterpart, we see that any path in a graph corresponds to a linear combination of vectors. Thus, following a certain path corresponds to adding together the vectors which represent the edges in the path. To visualize this type of vector addition, we will refer to the graph in Figure 4.1. If we follow the path v_1, v_4, v_2 by way of the edges e_1 and e_5, we start with vertex v_1 and end with vertex v_2. Since the graph is complete, this is the same as using the path v_1, v_2 along the edge e_3. If we view this relationship, using vectors from the vector space which correspond to the graph, we would expect that moving along a path would be equivalent to adding the vectors that correspond to the edges in the path. Thus, if we add the vectors representing the edges e_1 and e_5, we should get e_3. Since the addition of two vectors or (n+1)-tuples is performed by adding the vectors component-wise, we need the following rules in order to be able to add the components so that vector addition corresponds to moving along the edges of a path.

$$0 + 0 = 0$$

$$0 + 1 = 1$$

$$1 + 0 = 1$$

$$1 + 1 = 0$$

The last rule is the only one which differs from traditional addition, but we will soon see its importance.

Let us pause for a moment and look at a physical application of these rules for addition. Suppose that last night as you entered your garage through the door leading from the house, your little brother enters through the side door. Since the garage is completely dark, you both instinctively reach for the light switch which is located beside each door. The four addition rules represent the four possibilities of whether or not the garage light will come on. Since each rule is an equation, we can think of the numbers one and zero representing these actions.

$$\begin{pmatrix} \text{whether or not you} \\ \text{turn on your switch} \\ \text{"1" or "0"} \end{pmatrix} + \begin{pmatrix} \text{whether or not your little} \\ \text{brother turns on his switch} \\ \text{"1" or "0"} \end{pmatrix} = \text{result} \begin{pmatrix} \text{light is on} \\ \text{or it is off} \\ \text{"1" or "0"} \end{pmatrix}$$

Thus, the equation $0 + 0 = 0$ tells us that neither one of you moved your light switch; therefore, the garage light did not come on. Suppose that you heard your little brother come in and assumed that he would turn the light on. This represents the equation $0 + 1 = 1$ and the garage light comes on. However, it could have been the other way, your little brother could have heard you come in and expected you to turn on the light, thus the resulting equation would be 1 + 0 = 1. Now, what if both of you moved your switches to turn the light on? As you moved your switch to turn on the light your little brother moved his switch which caused the garage to remain in darkness. This equation is written as 1 + 1 = 0.

We now return to the idea of vector addition. An example of how the rules for adding components enables us to add vectors is seen when we add the vectors e_1 and e_5

$$e_1 + e_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 0+1 \\ 0+0 \\ 1+1 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_3 .$$

Graphically this means that if we follow the path v_1, v_4, v_2 (using the edges e_1 and e_5) in Figure 4.1, we will end up in the same place as we would have if we had taken the path v_1, v_2 along the edge e_3. Let us look at another example of vector addition.

$$e_3 + e_6 + e_8 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = e_1$$

This vector equation can be interpreted as saying that following the path v_1, v_2, v_3, v_4 (using the edges e_1, e_6 and e_8), is "equivalent" to going directly from vertex v_1 to vertex v_4 along edge e_1.

Now that we have a description of a vector in a vector space which represents a graph and know how vector addition works, we need to understand some of the other vector space properties. The zero vector, which is the 5-tuple whose entries are all zero, is a vector in this vector space, since it contains only an even number of ones.

To understand what the zero vector corresponds to in the graph we look at the following vector addition. We will let e_0 represent the zero vector.

$$e_3 + e_5 + e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_0$$

If we look at how these three edges are related in the graph, we see that they form the cycle v_1, v_2, v_5, v_1. Thus, we can see that the zero vector represents a cycle in the graph. Therefore, vectors which represent the edges of any cycle are linearly dependent. From our experience in linear algebra these vectors are linearly dependent, because there is a finite linear combination of edges whose sum is the zero vector and whose coefficients are not all zero. The linear combination can be written as

$1e_3 + 1e_5 + 1e_1 = e_0$. Since the coefficients are not all zero, the vectors which represent the edges e_3, e_5 and e_1 must be linearly dependent (as are any edges that form a cycle).

To determine the number of vectors in this vector space we recall that the entries in each 5-tuple consists of only the numbers zero and one. We know that there is only a finite number of ways that we can arrange the numbers zero and one in a 5-tuple so there will be an even number of ones in each tuple. The vector space corresponding to the graph G in Figure 4.1, consists of the vectors in the following set:

$$\left\{ \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix} \right\}.$$

It is important to notice that not all vectors in the vector space can be thought of as edges, since not all vectors contain exactly two ones. To see how these vectors are related to the vectors which represent edges, consider the following example of vector addition:

$$e_1 + e_7 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The resulting vector does not represent just a single edge, but the sum of two edges which are not incident to the same vertex.

An important property of any vector space is that it is closed with respect to the operations defined on it. Vector addition is closed, if given any two vectors in the vector space, their sum is again a vector in the vector space. The following exercise illustrates this idea.

Exercise 4.1

Prove that vector addition, in a vector space which corresponds to a graph, is closed. That is, using the rules of component-wise addition, prove that when adding any two vectors, each having an even number of ones, the resulting vector also has an even number of ones in the (n+1)-tuple.

The other operation in a vector space besides vector addition, called scalar multiplication, involves multiplying a vector by a *scalar*. For most purposes *scalars* are simply real or complex numbers. For the vector space corresponding to a graph, the **scalars** are either zero or one. Scalar multiplication involves multiplying the scalar by each component in the (n+1)-tuple (vector), using the usual multiplication rules for zero and one. Scalar multiplication is closed if, multiplying any given vector in the vector space by any scalar (zero or one), it is again a vector in the vector space.

Exercise 4.2

Prove that scalar multiplication, in a vector space which corresponds to a graph, is closed. That is, if any (n+1)-tuple, having an even number of ones, is multiplied by a scalar (either zero or one), the resulting vector has an even number of ones.

It can be shown that the requirements for a set to be a vector space (for example, commutativity of vector addition) are easily satisfied. Hence, G is indeed a vector space.

Unlike other non-trivial vector spaces that we have previously studied in linear algebra, the vector space corresponding to a graph contains only a finite number of vectors. Thus, there can only be a finite number of bases for the vector space. An understanding of how vector addition and scalar multiplication works, gives the necessary insight into how the vectors which represent the edges in a spanning tree form a basis for the vector space. For the vectors in a spanning tree to be a basis, we must show that these vectors span the vector space and are linearly independent.

To show that the vectors in a spanning tree span the vector space, we need to be able to represent every vector in the vector space using only these vectors. We will look at three cases.

Case 1

Consider any vector e in the vector space which represents an edge which joins two vertices v_i and v_j. Recall from the statement of Theorem 3 in

Section 3 that a spanning tree contains a unique path between any two vertices. Thus, there exists a unique path in the spanning tree between v_i and v_j and if we sum the vectors that represent the edges in this path, we will get the vector which represents the edge e.

Case 2

Consider any vector e in the vector space which has more than two ones. Recall that e can be thought of as representing the sum of edges which are not incident. For each of the edges in the sum which e represents, we apply Case 1 which gives a set of vectors from the spanning tree. When we sum these sets of vectors, we get the vector which represents e.

Case 3

The zero vector can be obtained by adding any vector, which represents an edge, in a spanning tree, to itself.

Moreover, the spanning tree contains no cycles, so it is impossible to find a linear combination of vectors which represent the edges of a spanning tree, with coefficients that are not all zero and whose sum is the zero vector. Therefore, the vectors corresponding to edges of the spanning tree are linearly independent. Because the vectors which represent the edges of the spanning tree are linearly independent and span the vector space, these vectors form a basis for the vector space. We will call a basis for a vector space corresponding to a graph an **edge basis**, if it consists entirely of vectors with only two ones in each n+1-tuple. These edge bases correspond to spanning trees. An example of a basis which is not an edge basis would be:

$$\left\{ \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix} \right\}.$$

A natural question to ask is exactly how may edge bases are there for a vector space that represents a given graph. Because an edge basis consists of the edges of a spanning tree, this question can be answered using the technique which was described in Section 2, which shows how to count the number of spanning trees in a graph. To calculate the number of spanning trees of the graph in Figure 4.1, we must first find the associated matrix. Since the graph G is complete, the matrix has no zeros and all of the diagonal entries are the same.

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$$

$$\text{and} \quad |D| = 125$$

Since the number of edge bases is the same as the number of spanning trees, we know that the graph G in Figure 4.1 has 125 edge bases.

Exercise 4.3

Given the graph G in Figure 4.2:

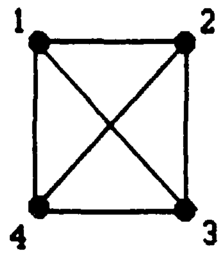


Figure 4.2

(a) Find all the vectors in the vector space corresponding to the graph G.

(b) Determine the number of edge bases in this vector space.

(c) Draw the spanning trees which correspond to the edge bases for this vector space.

Exercise 4.4

For each of the two Linear Algebra theorems listed below, write the statements of their Graph Theory counterparts.

(a) If S is a set which spans a finite dimensional vector space V, then there is a subset of S that is a basis for V.

(b) If S is a linearly independent set in a finite dimensional vector space V, then there is a basis for V which contains S.

Section 5 Transportation and Linear Programming Problems

To better show how to apply the knowledge we have gained about spanning trees to solve transportation problems, we first discuss a more general type of problem called a *linear programming problem*. Linear programming is a relatively new area in applied mathematics which arose in response to logistical problems which developed during World War II. Bland (1981) defines problems in linear programming as ones which are concerned with the distribution of scarce resources. To the government and corporate planner, linear programming has become a valuable tool in the decision making process and long-range planning. Linear programming covers a very broad range of problems. The Mr. Potatohead potato chip transportation problem of Section 1 is an example of the specific type of linear programming problem which we want to solve. Transportation problems like the Mr. Potatohead problem, are concerned with minimizing the cost of shipping a product from supply centers to demand centers. In this section we will investigate a technique to solve basic linear programming problems. Then in Section 6 we will build on this technique to create an efficient method to solve transportation problems.

It will be easier to understand the technique used to solve a basic linear programing problem if we first work an example. Consider the following problem: Momma Jane's is a small privately owned (by Jane) chocolate company which produces chocolate and almond snacks for a local airline company. During the winter months, Jane works alone and produces only two types of snacks; one type is a small box of candies made of milk chocolate with large whole almonds, and the other is a packet of specially seasoned almonds.

These airline snacks contain two main raw ingredients: milk chocolate and almonds. To ease her supply problems, Jane has recently negotiated a new contract with a bulk commodities co-operative to supply her with the needed raw ingredients. According to her new contract, Jane receives 80 ounces of milk chocolate and 128 ounces of whole almonds each day. To produce one box of the chocolates, 3 ounces of milk chocolate are needed along with 2 ounces of almonds. Jane puts 4 ounces of specially seasoned almonds in each almond packet. Through years of experience, Jane has found that it is always best to start with fresh chocolate and nuts each day. Not wishing to throw away her leftovers, she sells them to a nearby restaurant. She gets \$.10 per ounce for both the leftover milk chocolate and the seasoned almonds. Jane receives \$.54 for each box of the chocolates and \$.42 for each packet of almonds. Jane needs to know how many of each of these two snacks she should produce each day to maximize her revenue.

From experience, it has been found that writing a linear programming problem in a standard format will help to sort through and organize the great quantity of information found in each problem. We begin by identifying each of the variables in the problem.

Let: x_1 = number of boxes of milk chocolate with almonds that Jane will make

x_2 = number of packets of seasoned almonds that Jane will make

x_3 = ounces of leftover chocolate that Jane will sell

x_4 - ounces of leftover almonds that Jane will sell

Next, we create the **objective function,** $.54x_1 + .42x_2 + .10x_3 + .10x_4$, by summing the revenue generated by each variable. The objective function gives Jane's total revenue, determined in dollars. Jane's objective is to maximize this revenue:

$$\text{Maximize Revenue: } .54x_1+.42x_2+.10x_3+.10x_4.$$

To determine the number of snacks which Jane can produce each day, we must consider the constraint on each of the two ingredients (milk chocolate and almonds). Beginning with the ingredient milk chocolate, we must first determine the total amount available each day. We are told that each day Jane has available 80 ounces of milk chocolate. Next we consider where the chocolate is used: 3 ounces of chocolate is used per box (for a total of $3x_1$ ounces used for this type snack), and no chocolate is required in the packet of seasoned almonds (for a total of $0x_2$ ounces used in this type). Also, the amount of chocolate leftover at the end of the day is sold to the restaurant (for a total of x_3 ounces). The sum of these three quantities must equal the 80 ounces of available chocolate. In a similar way, the constraint equation for almonds can be found. Thus, we obtain the following constraint equations.

Chocolate constraint: $3x_1 + 0x_2 + x_3 \quad = 80$

Almond constraint: $2x_1 + 4x_2 \quad + x_4 = 128$

In addition, it is important to notice that each of the variables that we are working with only makes sense if its value is positive, that is, $x_i \geq 0$ $(i=1,2,3,4)$.

Recall that the goal of this problem is to maximize Jane's revenue. When we find the right combination of values which satisfy all the constraints and gives the maximum revenue possible, we say that this set of values is an **optimal solution**. We will use the following systematic procedure to determine values which satisfy the constraint equations. First, we need to find the column dimension of the matrix of constraint coefficients. In any matrix the row dimension and column dimension will always be equal. Thus, we need only determine one of these values. Suppose that the matrix of constraint coefficients has column dimension equal to m. Each column of the matrix of constraint coefficients contains coefficients for a specific variable and there is one column in the matrix for each variable. If we choose m linearly independent columns, set the variables not corresponding to these columns equal to zero, then solve for the remaining variables using the constraint equations, we will get a unique solution, called a **basic solution** in the m variables which correspond to the m columns that were chosen. When the value of each variable is greater than or equal to zero, the basic solution is called a **feasible basic solution**. With this in mind, we will need the following very important theorem from the theory of linear programming. A proof of this theorem can be found in Thie (1979, p. 99).

Theorem: If a linear programming problem has an optimal solution, then it has a feasible basic solution which is optimal.

The Theorem tells us that we can find the optimal solution by looking at feasible basic solutions. We may have to consider many solutions before we find the optimal solution. How will we know when we have found the optimal solution? We will look at two methods that can be used to determine which feasible basic solution is the optimal solution.

Method I The Exhaustive Search Method

Find all basic solutions by trying every combination of m columns that form a basis for the column space. Consider only feasible basic solutions, that is, solutions in which all values of the variables are greater than or equal to zero. Choose the feasible basic solution which gives the maximum or minimum (depending on the problem) value of the objective function. This will be the optimal solution.

This is a very straight-forward method, but not necessarily the quickest. Even with a small number of variables this method becomes quite cumbersome. Let us apply the Exhaustive Search Method to the Momma Jane's linear programming problem. The matrix of constraint coefficients is:

$$\begin{pmatrix} 3 & 0 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{pmatrix}$$

Since the column space has dimension 2, we may choose any two linearly independent columns of this matrix to find our first basic solution. Let us choose columns 1 and 2. This means that x_3 and x_4 are set equal to zero

and the constraint equations become:

Chocolate constraint: $3\,x_1 + 0\,x_2 = 80$

Almond constraint: $2\,x_1 + 4\,x_2 = 128$.

Solving this system of equations, we obtain $x_1 = \frac{80}{3}$, $x_2 = \frac{56}{3}$, $x_3 = 0$ and

$x_4 = 0$. Using these values, we find that Jane's revenue corresponding to

this feasible basic solution is:

$$.54x_1 + .42x_2 + .10x_3 + .10x_4 = .54(\frac{80}{3}) + .42(\frac{56}{3}) + .10(0) + .10(0)$$

$$= \$14.40 + \$7.84 = \$22.24$$

All feasible basic solutions for this problem are summarized below.

Columns Chosen	Solution	Profit
1 and 2	$x_1 = \frac{80}{3},\ x_2 = \frac{56}{3},\ x_3 = 0,\ x_4 = 0$	\$22.24
1 and 3	$x_1 = 64,\ x_2 = 0,\ x_3 = -112,\ x_4 = 0$ (not a feasible basic solution)	
1 and 4	$x_1 = \frac{80}{3},\ x_2 = 0,\ x_3 = 0,\ x_4 = 74.67$	\$21.87
2 and 3	$x_1 = 0,\ x_2 = 32,\ x_3 = 80,\ x_4 = 0$	\$21.44
2 and 4	not a basis for the column space	
3 and 4	$x_1 = 0,\ x_2 = 0,\ x_3 = 80,\ x_4 = 128$	\$20.80

We see that the optimal solution is found by using columns 1 and 2, giving a maximum profit of \$22.24. It is interesting to note that even though columns 1 and 3 are linearly independent, and thus form a basis for the column space, the solution of $x_3 = -112$ is negative so we do not have a feasible basic solution. It is also possible for two or more sets of columns to produce feasible basic solutions that when substituted into the objective function give the same optimal objective function value. In this case any of these feasible basic solutions is accepted as the optimal solution.

Exercises 5.1

Use the Exhaustive Search Method to solve the following linear programming problem: The Aloha Outerwear Company of Maui makes matching muumuus for ladies and traditional aloha style shirts for men. Each day they use a different pattern of material and must buy thread which matches. The pattern they will be sewing tomorrow, requires 1 spool of white, 3 spools of yellow and 4 spools of blue thread to sew a muumuu and 2 spools of both white and blue thread for each of the men's aloha shirts. The company makes a profit of $10 on each muumuu and $6 on each aloha shirt. They have available to use: 18 spools of white thread, 12 spools of yellow, and 24 spools of blue. Since they will not be sewing this pattern again for some time, they do not wish to store the leftover thread. They are able to sell it to a local retail store for $1 per spool for both white and yellow thread and $2 per spool for the blue. The president of Aloha Outerwear wishes to know how many muumuus and men's shirts need to be sewn to maximize tomorrow's profit. Hint: The following steps will help you solve the problem.

1) Define each of the variables.

Let: x_1 = number of muumuus sewn

x_2 = number of aloha shirts sewn

x_3 = boxes of leftover white thread sold

x_4 = boxes of leftover yellow thread sold

x_5 = boxes of leftover blue thread sold

2) Determine the objective function.

3) Write the constraint equation for each color of thread.

4) Summarize the information found by using the Exhaustive Search Method.

5) Pick out the optimal solution from the summary.

When using the Exhaustive Search Method, we must substitute *every* feasible basic solution into the objective function before we can find the optimal solution. As we have seen, this can become quite a long process. Thus, we are interested in finding another method which can be used to solve linear programming problems that does not require as many computations. The idea behind one such method is stated below:

Method II The Improvement Method

To use this method, first find a set of m linearly independent columns that yields a feasible basic solution. Then gradually "improve" on this set by systematically replacing one column in the set with another column, in such a manner that each new basic solution has a better objective function value. We continue this process until no improvement is possible, signifying that we have reached the optimal solution.

The statement of Method II is very general and it is not obvious at this point how to implement it. Actually there are several ways to accomplish the Improvement Method; one of these is the simplex method, which was introduced by George Dantzig in 1947. This efficient and versatile algorithm is largely responsible for the economic importance of linear programming. We will adapt some of the ideas behind the simplex method to solve transportation problems.

Section 6 Solving Transportation Problems

In this section we combine our knowledge of spanning trees and their relationship to edge bases with the insights that we have gained through the Exhaustive Search Method, to solve transportation problems using the Improvement Method.

To begin our study of transportation problems we consider the following transportation problem: The Pequot Lakes Wild Rice Company in central Minnesota recently merged with the Red Lake Rice Company in northern Minnesota and now has two warehouses. They sell most of their wild rice to a gourmet food chain with three outlets located in Fargo, North Dakota, and St. Paul and Duluth, Minnesota. Because of the especially early harvest this year, the warehouse in northern Minnesota has 10 cases of wild rice ready to be shipped and the central Minnesota warehouse has 15 cases. The shipping agent for the gourmet food company has ordered 4 cases of rice for the Fargo store, 14 cases for the St. Paul store and 7 cases for Duluth. She wants to know how many cases to request from each warehouse so that her company pays the lowest shipping cost. The shipping routes in Figure 6.1 are labeled to reflect the shipping costs in dollars per case.

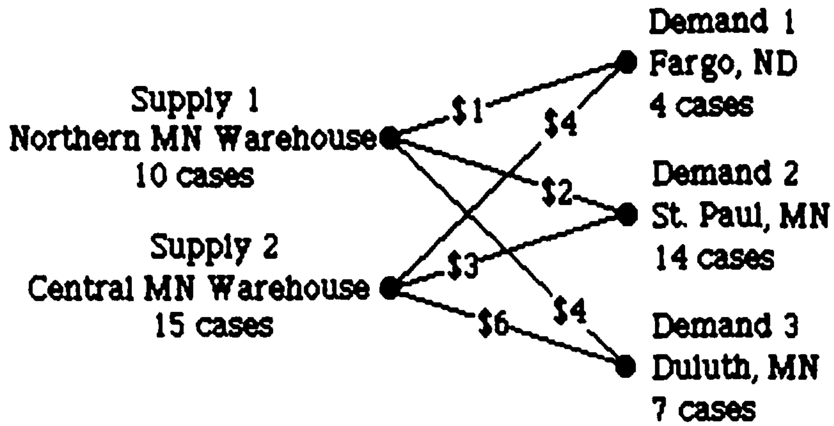


Figure 6.1

We will use the following convenient notation to represent the shipment quantities. Let x_{ij} be the amount shipped from supply i to demand j, where $i=1, 2$ and $j=1, 2, 3$. The standard format of objective function and constraint equations which was used to write linear programming problems is given below.

Minimize Cost: $1x_{11} + 2x_{12} + 4x_{13} + 4x_{21} + 3x_{22} + 6x_{23}$

$$
\begin{array}{llllllll}
\text{Subject to:} & x_{11} + & x_{12} + & x_{13} & & & & = 10 \quad \text{Supply 1} \\
 & & & & x_{21} + & x_{22} + & x_{23} & = 15 \quad \text{Supply 2} \\
 & x_{11} & & & + x_{21} & & & = 4 \quad \text{Demand 1} \\
 & & x_{12} & & & + x_{22} & & = 14 \quad \text{Demand 2} \\
 & & & x_{13} & & & + x_{23} & = 7 \quad \text{Demand 3}
\end{array}
$$

and all $x_{ij} \geq 0$.

If we were to use the Exhaustive Search Method, we would need to consider the column space for the coefficient matrix of the constraint equations shown below.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Taking a careful look at this matrix, we see that each column has exactly two ones. This will always be true when we are dealing with a transportation problem. In this matrix, each column relates to an edge in the bipartite graph which represents one particular shipping route, and each edge of this graph is represented by a column in the matrix. To confirm this observation, the graph representing the transportation problem is given in Figure 6.2, with the vertices numbered to reflect supply as vertices 1 and 2 and demand as vertices 3, 4 and 5.

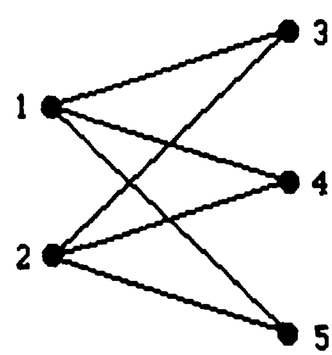


Figure 6.2

The Exhaustive Search Method requires that we choose a basis for the column space. Because each column corresponds to an edge of the complete bipartite graph, doing so is equivalent to choosing a spanning tree of that graph. (Actually we have shown this only when we use the rules given on page 82. It remains true when we allow use of real coefficients other than 0/1 and the usual real arithmetic operations for linear combinations; using alternating coefficients of -1 and +1 for the edges in a cycle shows that these edges are linearly dependent.) However, some spanning trees require a negative amount to be shipped, similar to the situation in the Momma Jane's chocolate problem when columns 1 and 3 were chosen. To understand how this can happen we will consider two spanning trees.

Spanning Tree 1 All amounts shipped are positive.
Figure 6.3 displays a spanning tree with the number of cases of wild rice shipped written on each edge.

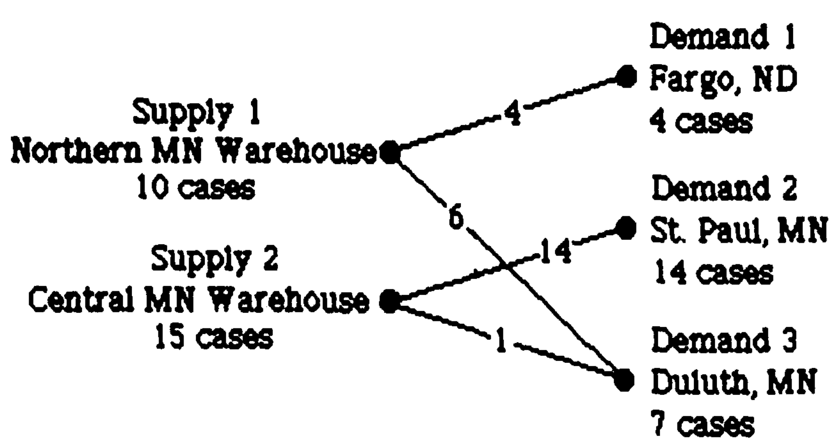


Figure 6.3

We will consider three different methods to determine the number of cases of wild rice which should be shipped from each warehouse.

Method 1

The edges in the graph of Figure 6.3 represent the variables x_{11}, x_{13}, x_{22}, x_{23}. If we choose the corresponding columns from the coefficient matrix, the augmented matrix becomes

$$\begin{array}{c} \begin{array}{cccc} x_{11} & x_{13} & x_{22} & x_{23} \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 & 15 \\ 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 14 \\ 0 & 1 & 0 & 1 & 7 \end{array}\right). \end{array}$$

Reducing this matrix so the coefficient matrix of the reduced system becomes the identity matrix, we get

$$\begin{array}{c} \begin{array}{cccc} x_{11} & x_{13} & x_{22} & x_{23} \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 14 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \end{array}$$

We see that the solution to this system, $x_{11} = 4$, $x_{13} = 6$, $x_{22} = 14$, $x_{23} = 1$, is exactly the number of cases of rice indicated on each edge in Figure 6.3.

Method 2

The original system in Method 1 was very easy to solve. In fact, if we looked at the system carefully we would be able to determine several values without doing any work. We could then substitute these values into other equations and find the rest of the solution values. This procedure should remind us of the technique used to solve systems of linear equations in upper triangular form, called back substitution. If we start again with the matrix

$$\begin{array}{cccccc} x_{11} & x_{13} & x_{22} & x_{23} & & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 & 15 \\ 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 14 \\ 0 & 1 & 0 & 1 & 7 \end{array}\right)$$

which has more equations than unknowns, we will end up with a row of zeros when the matrix is reduced (as seen in Method 1). Thus, we may delete any row in the augmented matrix which is a linear combination of other rows in the matrix, before we start the row reduction process. We choose to delete the last row in this matrix, since it can be shown that it is a linear combination of the other rows. If we rearrange the columns of the new matrix, being careful to keep track of the variables which correspond to the column, we can create the upper triangular coefficient matrix given below.

$$\begin{array}{cccc}
x_{13} & x_{23} & x_{11} & x_{22}
\end{array}$$
$$\left(\begin{array}{cccc|c}
1 & 0 & 1 & 0 & 10 \\
0 & 1 & 0 & 1 & 15 \\
0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 1 & 14
\end{array}\right)$$

Using back substitution to solve for the variables, we find that $x_{11} = 4$, $x_{13} = 6$, $x_{22} = 14$, and $x_{23} = 1$, which are the number of cases of wild rice that need to be shipped from the warehouses to the respective stores.

<u>Method 3</u>

The final method describes in words what was being done in Methods 1 and 2 using Linear Algebra. Recall that by Lemma 2 of Section 3, every spanning tree has at least one vertex that has degree 1. For example, the vertex representing the gourmet store in Fargo, ND that requires 4 cases of wild rice, has degree 1. Since the northern warehouse is the lone supplier to Fargo, this forces 4 of the 10 cases of rice at the northern warehouse to be shipped to Fargo. This is the physical interpretation of the third line in the matrix equation from Method 2 which is repeated below.

$$\begin{array}{cccc}
x_{13} & x_{23} & x_{11} & x_{22}
\end{array}$$
$$\left(\begin{array}{cccc|c}
1 & 0 & 1 & 0 & 10 \\
0 & 1 & 0 & 1 & 15 \\
0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 1 & 14
\end{array}\right)$$

There are 6 cases remaining at the warehouse and they can be either shipped to St. Paul or Duluth. However, since the shipping route connecting the northern warehouse to St. Paul has been excluded, all 6 cases are shipped to Duluth. This is the same as using back substitution in the first equation of the augmented matrix above. The Duluth store needs 1 more case of rice to fulfill their requirement of 7 and it must come from the central Minnesota warehouse. This is how we interpret the second equation in the matrix above. St. Paul requires 14 cases and since the central warehouse has 14 cases left, they are all shipped to St. Paul. In the above matrix, this can be seen using the last equation.

Spanning Tree 2 All amounts shipped are negative.
Figure 6.4 displays a spanning tree with the number of cases of wild rice shipped from the warehouse written on each edge. Note that one edge has a negative amount to be shipped.

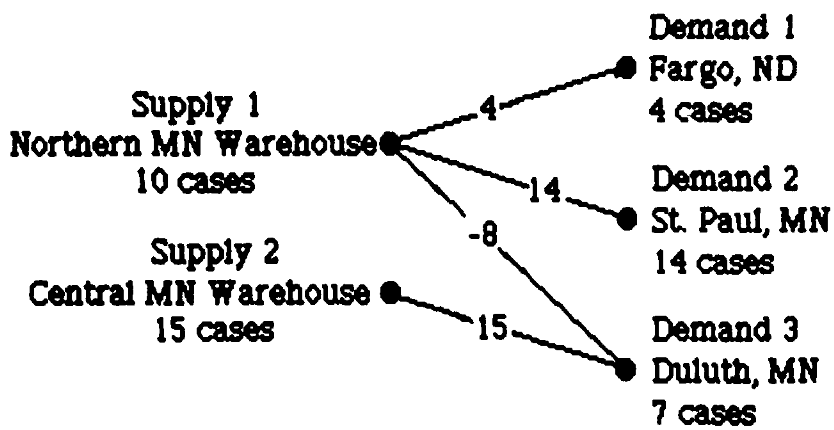


Figure 6.4

Exercise 6.1

(a) Use the technique of Method 1 in Spanning Tree 1 to confirm the number of cases of rice to be shipped from the warehouses to the respective stores as indicated in Figure 6.4.

(b) Use the technique of Method 2 in Spanning Tree 1 to confirm the answers you found in part (a).

Method 3

Again we apply Lemma 2 of Section 3, (every spanning tree has at least one vertex that has degree 1) to the graph in Figure 6.4 and begin by considering where the central warehouse can ship its cases of rice. Since there is only one shipping route available, all 15 cases must be shipped to Duluth. However, only 7 cases are needed at the store in Duluth, so the remaining 8 cases must be shipped on to the northern warehouse. The additional time required to ship the 8 cases from the central warehouse to the northern warehouse through Duluth would eliminate this as a possible shipping route. This is the physical interpretation of -8 in Figure 6.4. Since the solution will contain a negative number, we do not have a feasible basic solution and there is no need to find the other values in this solution.

If we were going to use the Exhaustive Search Method to minimize the shipping cost, we would have to check both the spanning trees in Figures 6.3 and 6.4 as part of the solution technique. However, we plan on using the Improvement Method to solve this transportation problem. We need to be able to select a spanning tree, which does not contain any negative shipping quantities, from the graph representing the transportation problem. Even though we have already found such a spanning tree for this problem (Figure 6.3), we need a procedure to find one in general. We begin by drawing the graph of a transportation problem in the following standard format.

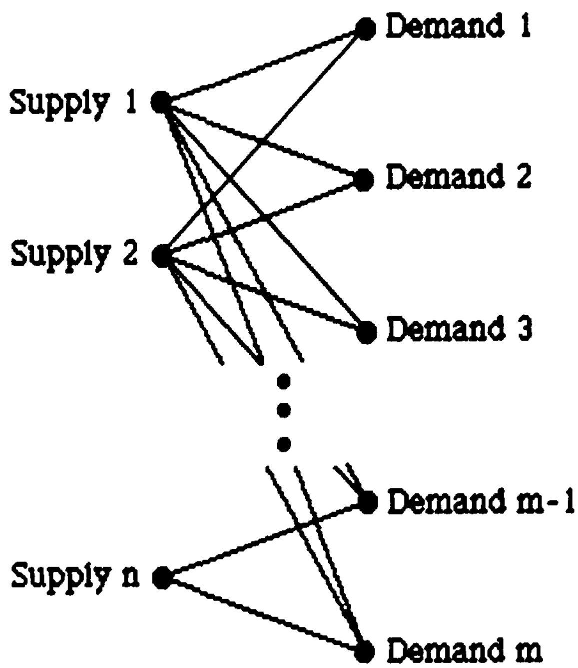


Figure 6.5

Northwest Rule

Given a complete bipartite graph drawn using the standard format of Figure 6.5, begin with the northwest supply center (Supply 1) and if possible, fill the order of Demand 1. Indicate the quantity shipped on the line connecting the supply center with the demand center. Then, continue filling demand centers orders until the product at Supply 1 has been depleted. Repeat this process with Supply 2. If Supply 1 was not able to fill the order of Demand 1, then Supply 2 continues to fill the order. This procedure continues until all supply centers have been depleted and all demand centers requests have been filled.

If we use the Northwest Rule, the resulting spanning tree will only have positive quantities designated to be shipped from supply centers to demand centers. Figure 6.6 shows the steps involved in applying the Northwest Rule to the Pequot Lakes Wild Rice problem. Notice that in this problem the total supply equals the total demand, so all items are shipped. We consider this condition to be true for all our transportation problems.

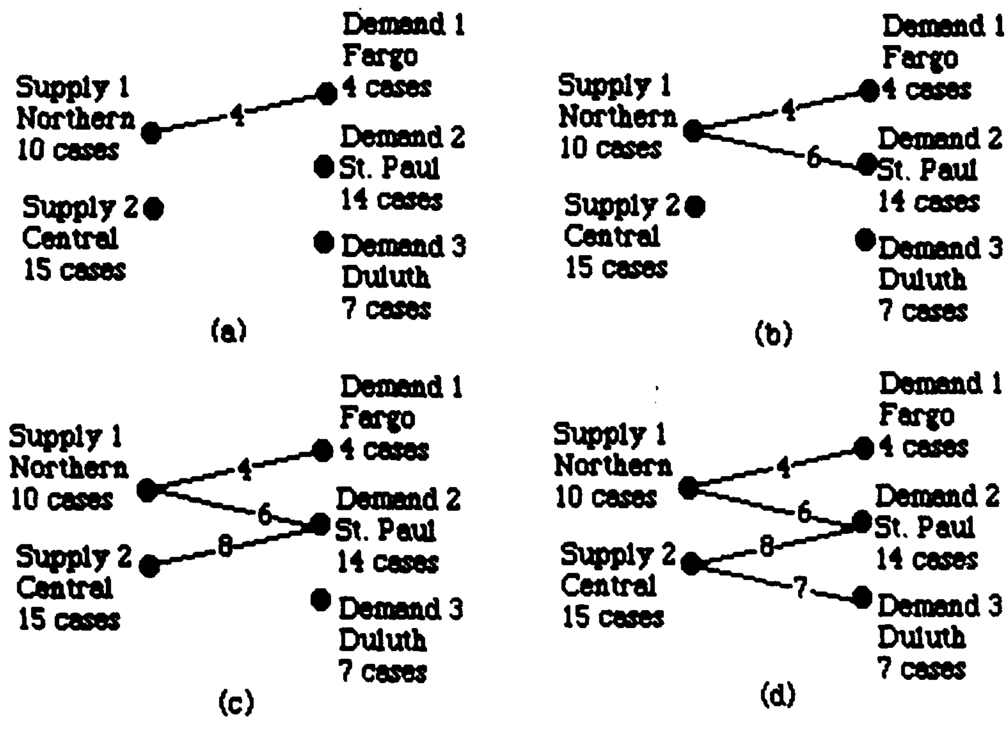


Figure 6.6

We will apply the Improvement Method to the spanning tree in Figure 6.6(d). The cost that we will try to improve (minimize) is:

$$1x_{11} + 2x_{12} + 4x_{13} + 4x_{21} + 3x_{22} + 6x_{23}$$

$$= 1\,(4) + 2\,(6) + 4\,(0) + 4\,(0) + 3\,(8) + 6\,(7)$$

$$= \$82.$$

The general goal of the Improvement Method is to find an alternative spanning tree which will give a cost less than $82. To do this we will attempt to add an edge to the spanning tree which will create a cycle, then take away an edge on the cycle, leaving a different spanning tree with a cost less than $82. From the spanning tree in Figure 6.6(d) we notice that there are only two possible edges that we can add. They are the ones representing the shipping routes from the northern warehouse (Supply 1) to Duluth (Demand 3) and from the central warehouse (Supply 2) to Fargo (Demand 1).

<u>First, we determine if adding the shipping route from the central warehouse to Fargo will lower (improve) the shipping cost.</u>

From Figure 6.1, we see that the cost to ship a case of wild rice from the central warehouse directly to Fargo is $4. Since Figure 6.6(d) is a spanning tree, Theorem 3 of Section 3 (any pair of vertices in a tree is joined by exactly one path) tells us that the path from the central warehouse to Fargo is unique. We want to check to see if shipping a case of rice along the unique path in Figure 6.6(d) is more cost effective than shipping it directly from the central warehouse to Fargo. The unique shipping route from Figure 6.6(d) which is used to move one case of rice from the central warehouse to Fargo starts at the central warehouse, goes to St. Paul, back to the northern warehouse, and then on to Fargo. The equivalent cost is

$$c_{22} - c_{12} + c_{11} = 3 - 2 + 1 = 2,$$

where c_{ij} is the cost to ship one unit along the edge x_{ij}. The cost of c_{12} is -2 because it is equivalent to shipping one less unit from Supply 1 to Demand 2, thus decreasing the cost. That is, we are saving \$2. Since the cost is less to ship a case from the central warehouse to Fargo, in the spanning tree of Figure 6.6(d), than it is to ship it directly, we decide not to add this shipping route to our tree. We now consider adding the only other possible shipping route.

<u>Determine if adding the shipping route from the northern warehouse to Duluth will lower (improve) the shipping cost.</u>

From Figure 6.1, the cost to ship a case of wild rice from the northern warehouse directly to Duluth is \$4. The equivalent shipping route in the spanning tree of Figure 6.6(d) would be to ship the case from the northern warehouse to St. Paul, back to the central warehouse, and then to Duluth. The cost is:

$$c_{12} - c_{22} + c_{23} = 2 - 3 + 6 = \$5.$$

Since the cost of shipping direct is cheaper, we add the edge which represents the shipping route from the northern warehouse to Duluth. The newly added edge is indicated in Figure 6.7 by the bold line.

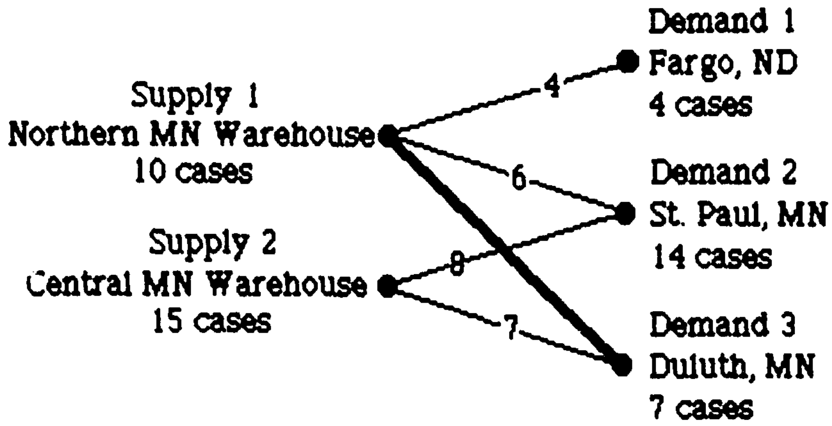


Figure 6.7

Since the unique path from the northern warehouse to Duluth of Figure 6.6(d) and the newly added edge that goes directly from the northern warehouse to Duluth are both included in Figure 6.7, we now have a cycle. Therefore, we must decide which edge of the cycle is to be removed so that we once again have a spanning tree. Since it is cheaper to ship directly from the northern warehouse to Duluth, we want to determine how many cases of rice can be rerouted. Figures 6.8(a)-(f) shows what happens as we reroute one case at a time along the newly added shipping route.

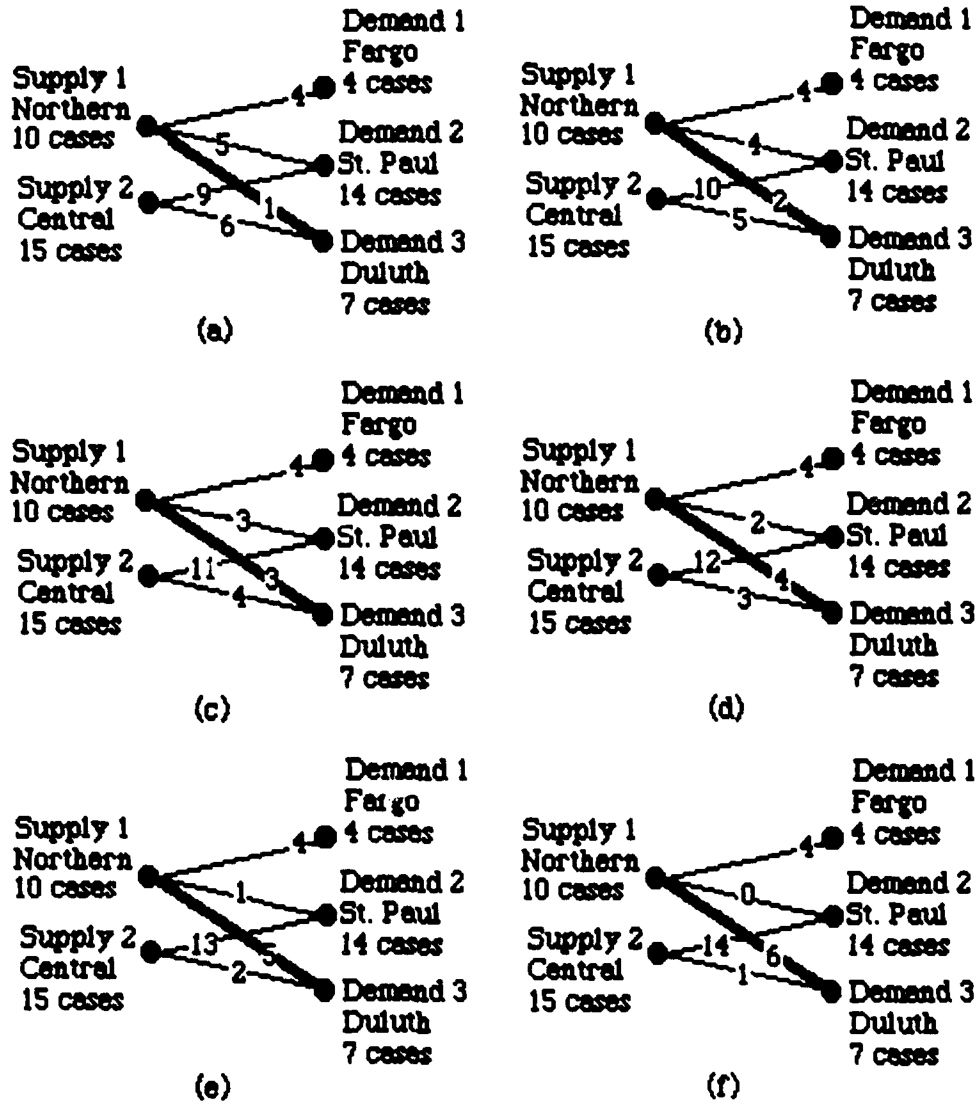


Figure 6.8

From Figure 6.8(f) we see that the maximum number of cases which can be rerouted is six. This constraint is determined by the shipping route from the northern warehouse to St. Paul because *it is the first route in the cycle* that becomes zero. As a result this is the edge that is dropped. When this is done, we have the spanning tree in Figure 6.9.

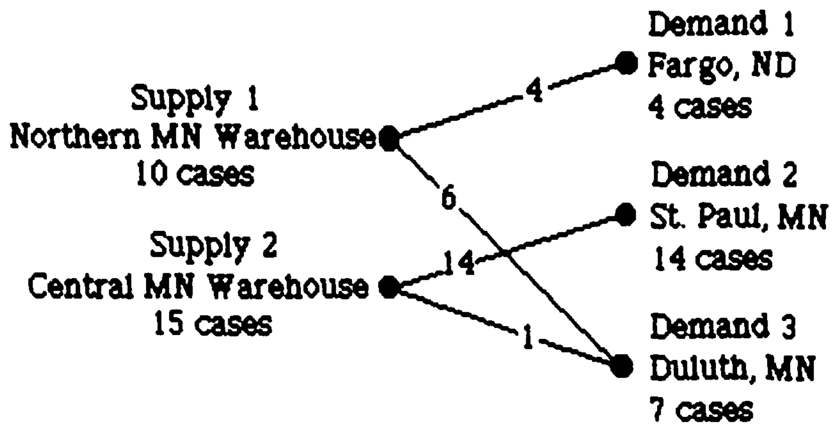


Figure 6.9

Have we found the optimal solution? To determine if we have found the optimal solution, we must repeat the procedure above. That is, we must check all the possibilities for new shipping routes. The only two shipping routes that we can consider adding, would be the one from the central warehouse to Fargo and the one from the northern warehouse to St. Paul. Shipping a case of wild rice from the central warehouse to Fargo, costs $4 a case. The equivalent cost to ship one case of wild rice from the central warehouse to Fargo in the spanning tree of Figure 6.9 would be to ship the case from the central warehouse to Duluth, back to the northern warehouse, and then to Fargo. The cost would be:

$$c_{23} - c_{22} + c_{11} = 6 - 4 + 1 = \$3.$$

Since it is cheaper to ship a case from the central warehouse to Fargo, in the spanning tree of Figure 6.9, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Shipping a case of wild rice from the northern warehouse to St. Paul, costs $2 a case. The equivalent cost to ship one case of wild rice from the northern warehouse to St. Paul in the spanning tree of Figure 6.9, would be to ship the case from the northern warehouse to Duluth, back to the central warehouse, and then to St. Paul. The cost would be:

$$c_{13} - c_{23} + c_{22} = 4 - 6 + 3 = \$1.$$

Since the cost is less to ship a case from the northern warehouse to St. Paul, in the spanning tree of Figure 6.9, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. We conclude that the optimal solution is represented by the graph in Figure 6.9. The minimum cost is:

$$1x_{11} + 2x_{12} + 4x_{13} + 4x_{21} + 3x_{22} + 6x_{23}$$

$$= 1\,(4) + 2\,(0) + 4\,(6) + 4\,(0) + 3\,(14) + 6\,(1)$$

$$= \$76.$$

Exercise 6.2

The graph in Figure 6.10 represents a transportation problem where the shipping cost per item is indicated on each edge in the graph.

(a) Write the objective function and constraint equations for this problem.

(b) Use the Northwest Rule to find a spanning tree which represents a feasible basic solution.

(c) Use the Improvement Method to find the optimal solution and show that no matter which edge you consider adding to the graph first, both give the same optimal solution.

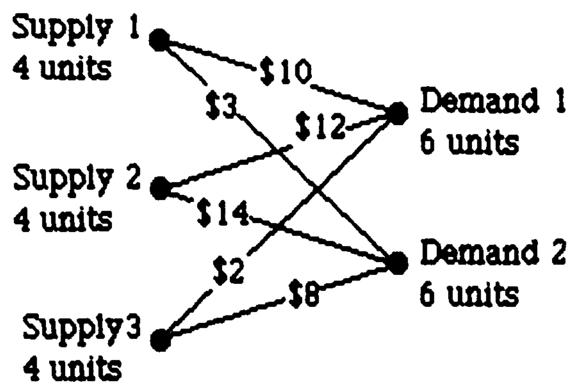


Figure 6.10

Exercise 6.3

Use the Improvement Method to solve the Mr. Potatohead Potato Chip Company transportation problem of Section 1. (Be sure to include the objective function and constraint equations for this problem.)

Section 7 Summary

The overall goal of this module was to illustrate how to take advantage of interrelationships between Graph Theory and Linear Algebra to solve a transportation problem. However, we had a lot of preliminary work to do before we could consider this type of problem. First we needed to understand what a graph was, so we began by introducing some basic definitions and ideas of Graph Theory. This led us to the concept of a spanning tree and an algorithm to determine the number of spanning trees in a graph. Before we could use this new information about Graph Theory, we needed to study theorems to show us how these ideas were related to each other. Next we looked at how this new information paralleled definitions and theorems we had already studied in Linear Algebra. Of particular interest was the relationship between a graph and the vector space which represents it. We also saw how a linear programming problem could be solved by finding all of the basic feasible solutions, then choosing an optimal solution from among them. Although this technique required many calculations, we were able to modify it by considering a basic feasible solution and improving upon it until an optimal solution was reached. Finally, we were able to use this knowledge to solve a transportation problem.

References

Harary, F. (1969). Graph theory. Reading, MA: Addison-Wesley.

Thie, P. (1979). An introduction to linear programming and game theory. New York: John Wiley & Sons.

Bland, R. G. (1981). The allocation of resources by linear programming. Scientific American, 244(18), 126-144.

Application II Appendix: Solutions to Exercises

Exercise 1.1

(a) The degree of v_3 is 4.

The degree of v_4 is 3.

The degree of v_5 is 2.

(b) One possible walk that is not a path is

$$v_2, v_3, v_1, v_4, v_3, v_4, v_5.$$

(c) The path created from the above walk is

v_2, v_3, v_1, v_4, v_5 or v_2, v_3, v_5 or v_2, v_1, v_3, v_4, v_5 or v_2, v_3, v_4, v_5

(d) One possible cycle with 3 distinct vertices is

$$v_3, v_5, v_4, v_3.$$

One possible cycle with 4 distinct vertices is

$$v_1, v_2, v_3, v_4, v_1.$$

Exercise 1.2

(a) Figure A.1 is a connected graph.

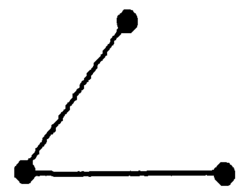


Figure A.1

(b) Figure A.2 is a disconnected graph.

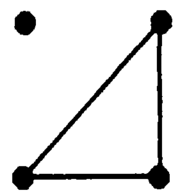


Figure A.2

(c) Figure A.3 is a complete graph with 4 vertices.

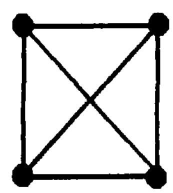


Figure A.3

(d) No, it is not possible to have a complete disconnected graph.

Exercise 1.3

Since G is connected, we know that between any two vertices in G there is a walk. We will use that walk to find a path between the vertices. Choose any two distinct vertices, for example, u and v in G. Then there exists a walk from u to v. If this walk is a path, then we are done. If not, let w be the first vertex which is repeated in the walk. Delete from the walk all vertices and edges which immediately follows the first time that w occurs, up to and including the second time w occurs. We notice that this is still a walk from u to v in the graph G. If this new walk is a path, then we are done. If not, then we continue to use the procedure above until no vertex is repeated. This yields a walk from u to v which is a path. Because the starting and ending vertices were chosen arbitrarily, we have shown that there is a path between every two distinct vertices. □

<u>Exercise 1.4</u>

(a) A subgraph of the graph in Figure 1.2, which is not a spanning subgraph, is given in Figure A.4.

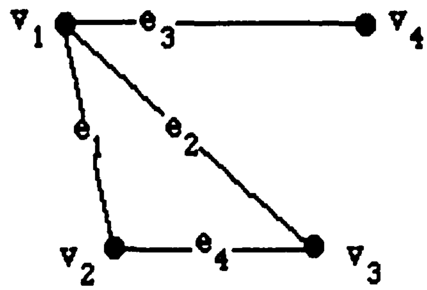


Figure A.4

(b) A spanning subgraph of the graph in Figure 1.2 is given in Figure A.5.

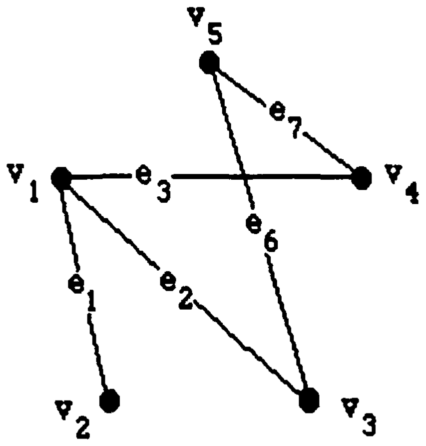


Figure A.5

Exercise 2.1

(a) A tree of the graph in Figure 1.2, which is not a spanning tree, is given in Figure A.6.

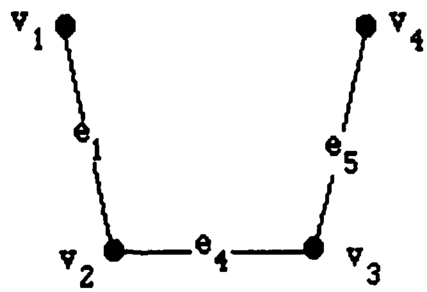


Figure A.6

(b) No, not all trees are spanning trees.

(c) Yes, all spanning trees are trees.

Exercise 2.2

The graph G has eight spanning trees. They are given in Figure 2.3.

Exercise 2.3

Observe that n=3.

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

The number of spanning trees is $|D| = 3$.

The three spanning trees of G are given in Figure A.7.

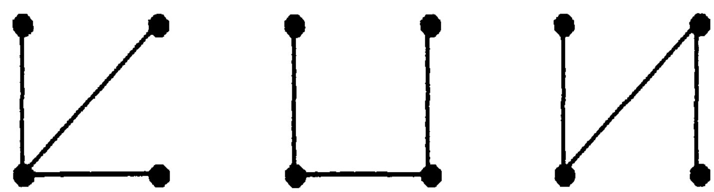


Figure A.7

Exercise 2.4

Observe that n=4.

$$A = \begin{pmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & -1 & -1 \\ 0 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

The number of spanning trees is $|D| = 12$.

The twelve spanning trees of G are given in Figure A.8.

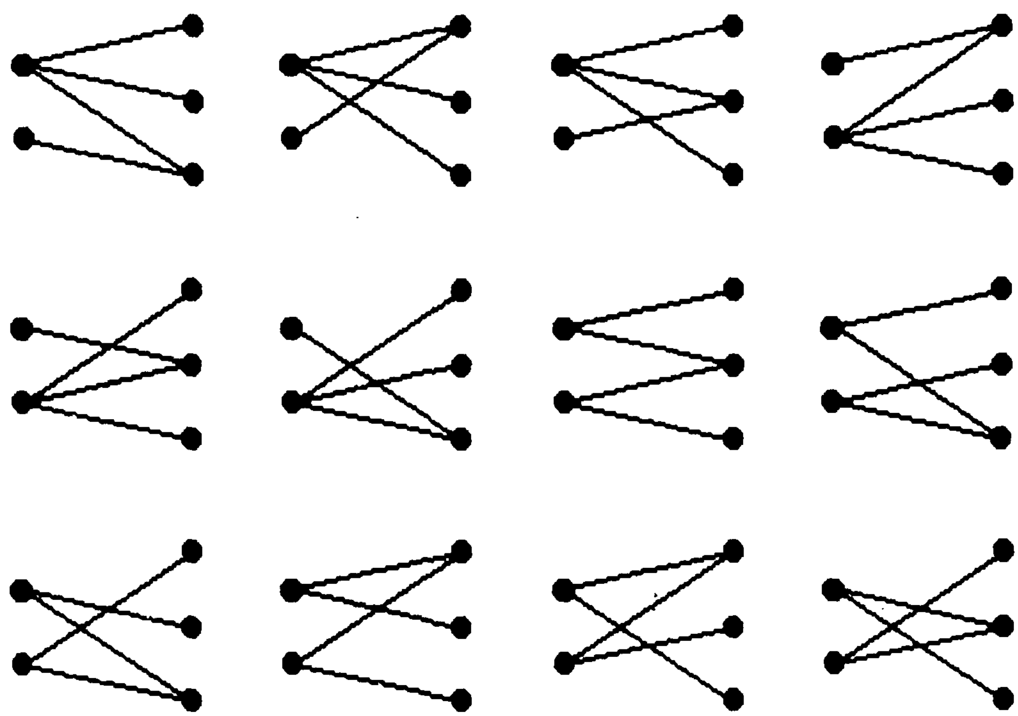


Figure A.8

<u>Exercise 4.1</u>

We must show that if we add any two vectors which have an even number of entries that are ones, we will get another vector which has an even number of ones as entries.

Let x and y be two vectors with an even number of ones in their tuples.

<u>Case 1</u> $x = y$.

Then the resulting vector $x+y$ is the zero vector (all of whom entries are zero), which has and even number of ones.

<u>Case 2</u> Either x or y is the zero vector, but not both.

If x were the zero vector, then $x+y = y$, which has an even number of ones.

If y were the zero vector, then $x+y = x$, which has an even number of ones.

<u>Case 3</u> Neither x nor y are zero, nor are they equal.

<u>Subcase 1</u> x and y do not have a one in the same position in each vector.

Then $x+y$ will have (number of ones in x)+(number of ones in y) and the sum of two even numbers in an even number.

<u>Subcase 2</u> There is an odd number of corresponding positions in x and y that contain ones.

Then there is an odd number of ones in x that do not have a one in the corresponding position in y. Similarly, there is an odd number of ones in y that do not have a one in the corresponding position in x. Thus, there is (odd number of ones in x)+(odd number of ones in y) or an even number of ones in $x+y$.

<u>Subcase 3</u> There is an even number of corresponding positions in x and y that contain ones.

Then there is an even number of ones in x that do not have a one in the corresponding position in y. Similarly, there is an even number of ones in y that do not have a one in the corresponding position in x. Thus, there is (even number of ones in x)+(even number of ones in y) or an even number of ones in $x+y$.

Therefore vector addition is closed.

Exercise 4.2

We must show that if we multiply a vector with an even number of entries that are one by any scalar, then we will again have a vector with an even number of ones.

Let x be any vector with an even number of entries that are one.

Case 1 Multiply x by the scalar 0.

When we multiply by the scalar 0, we multiply each entry in the tuple by zero. The resulting vector is the zero vector, which has an even number of ones.

Case 2 Multiply x by the scalar 1.

When we multiply by the scalar 1, we multiply each entry in the tuple by one. The resulting vector is x, the vector we started with, which has and even number of ones.

Therefore scalar multiplication is closed.

Exercise 4.3

(1) To find the vectors in the vector space corresponding to the graph G in Figure 4.2, we notice that n=3, so the vectors will be 4-tuples, all of whose entries are zeros and ones, and only an even number of ones in each tuple. Thus, the vector space is:

$$\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \right\}$$

(2) We can calculate the number of spanning trees by using the algorithm found in Section 2. Since we are dealing with a complete graph, the associated matrix has no zeros and all of the diagonal entries are the same.

$$A = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$\text{and } \quad |D| - 16$$

Because the number of edge bases is the same as the number of spanning trees, we know that the graph G has 16 edge bases. The spanning trees, which are the graphs corresponding to these edge bases, are given in Figure A.9.

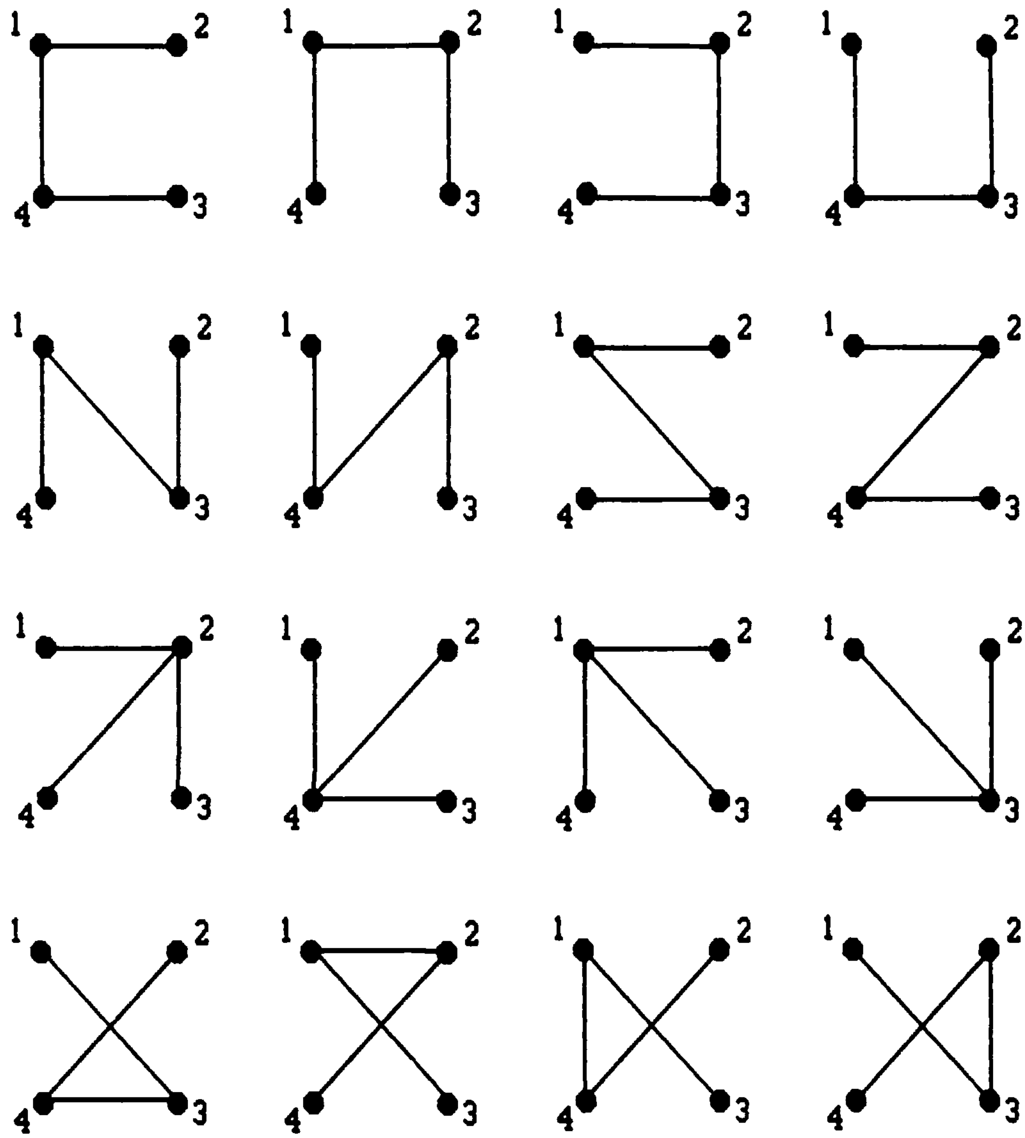


Figure A.9

Exercise 4.4

(a) If G is a connected graph, then there exists a spanning tree for G.

(b) If S is a set of edges from a connected graph G, such that S has no cycles, then there is a spanning tree of G which contains S.

<u>Exercise 5.1</u>

Maximize Profit: $10\,x_1 + 6\,x_2 + 1\,x_3 + x_4 + 2\,x_5$

Subject to:

$$
\begin{aligned}
1\,x_1 + 2\,x_2 + x_3 \qquad\qquad &= 18 \\
3\,x_1 + 0\,x_2 \qquad + x_4 \qquad &= 12 \\
4\,x_1 + 2\,x_2 \qquad\qquad + x_5 &= 24
\end{aligned}
$$

The column space has dimension 3. All basic solutions for this problem are summarized below.

Columns	Solution	Profit
1, 2, 3	$x_1=4,\ x_2=4,\ x_3=6,\ x_4=0,\ x_5=0$	\$70
1, 2, 4	$x_1=2,\ x_2=8,\ x_3=0,\ x_4=6,\ x_5=0$	\$74
1, 2, 5	$x_1=4,\ x_2=7,\ x_3=0,\ x_4=0,\ x_5=-6$	impossible
1, 3, 4	$x_1=6,\ x_2=0,\ x_3=12,\ x_4=-6,\ x_5=0$	impossible
1, 3, 5	$x_1=4,\ x_2=0,\ x_3=14,\ x_4=0,\ x_5=8$	\$70
1, 4, 5	$x_1=18,\ x_2=0,\ x_3=0,\ x_4=-42,\ x_5=-48$	impossible
2, 3, 4	$x_1=0,\ x_2=12,\ x_3=-6,\ x_4=12,\ x_5=0$	impossible
2, 3, 5	not a basis for the column space	
2, 4, 5	$x_1=0,\ x_2=9,\ x_3=0,\ x_4=12,\ x_5=6$	\$78
3, 4, 5	$x_1=0,\ x_2=0,\ x_3=18,\ x_4=12,\ x_5=24$	\$78

By examining these feasible basic solutions, we see that there are two feasible basic solutions which produce the same optimal solution. Thus, the maximum profit is \$78.

<u>Exercise 6.1</u>

(a) The edges in the graph of Figure 6.4 represent the variables x_{11}, x_{13}, x_{22}, x_{23}. If we choose the corresponding columns from the coefficient matrix, the augmented matrix becomes

$$\begin{array}{c}
\begin{array}{cccc} x_{11} & x_{12} & x_{13} & x_{23} \end{array} \\
\left(\begin{array}{cccc|c}
1 & 1 & 1 & 0 & 10 \\
0 & 0 & 0 & 1 & 15 \\
1 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 14 \\
0 & 0 & 1 & 1 & 7
\end{array}\right).
\end{array}$$

Reducing this matrix, we get

$$\begin{array}{c}
\begin{array}{cccc} x_{11} & x_{12} & x_{13} & x_{23} \end{array} \\
\left(\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 14 \\
0 & 0 & 1 & 0 & -8 \\
0 & 0 & 0 & 1 & 15 \\
0 & 0 & 0 & 0 & 0
\end{array}\right).
\end{array}$$

We see that the solutions to this matrix, $x_{11}= 4$, $x_{12}= 14$, $x_{13}= -8$, $x_{23}= 15$, are exactly the number of cases of rice indicated on each edge in Figure 6.4.

(b) Since the augmented matrix used in part (a) has more equations than unknowns, we will end up with a row of zeros when the matrix is reduced (as seen in part (a)). Thus, we may delete any row in the augmented matrix which is a linear combination of other rows in the matrix, before we start the row reduction process. We choose to delete the last row in the augmented matrix, because it can be shown to be a linear combination of the other rows. If we carefully rearrange the columns of the new matrix, we can create the upper triangular coefficient matrix given below.

$$\begin{array}{c} \begin{array}{cccc} x_{13} & x_{23} & x_{11} & x_{12} \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 0 & 15 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 14 \end{array}\right) \end{array}$$

Using back substitution to solve for the variables, we find that $x_{11} = 4$, $x_{12} = 14$, $x_{13} = -8$, and $x_{23} = 15$, which are the number of cases of wild rice that need to be shipped from the warehouses to the respective stores.

Exercise 6.2

(a) Before we can use the Improvement Method, we need the objective function and the system of constraint equations for the transportation problem.

Minimize: $10x_{11}+3x_{12}+12x_{21}+14x_{22}+2x_{31}+8x_{32}$

Subject to:

$$
\begin{array}{lllllllll}
x_{11} & + x_{12} & & & & & = 4 & & \text{Supply 1} \\
& & x_{21} & + x_{22} & & & = 4 & & \text{Supply 2} \\
& & & & x_{31} & + x_{32} & = 4 & & \text{Supply 3} \\
x_{11} & & + x_{21} & & & & = 6 & & \text{Demand 1} \\
& & & x_{22} & & + x_{32} & = 6 & & \text{Demand 2}
\end{array}
$$

and all $x_{ij} \geq 0$.

(b) Using the Northwest Rule, we find the spanning tree in Figure A.10 which represents a basic feasible solution.

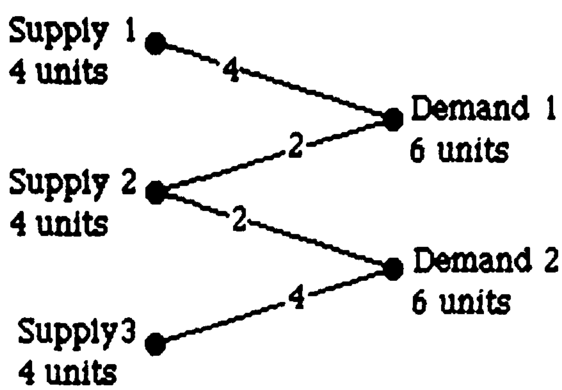


Figure A.10

(c) The amount that we will try to improve (minimize) is

$$10x_{11} + 3x_{12} + 12x_{21} + 14x_{22} + 2x_{31} + 8x_{32}$$

$$= 10\,(4) + 3\,(0) + 12\,(2) + 14\,(2) + 2\,(0) + 8\,(4)$$

$$= \$124$$

From the spanning tree we see that there are only two possible edges that we can consider adding that may improve on the cost of $124. They are the route from Supply 1 to Demand 2 and the route from Supply 3 to Demand 1.

<u>Determine if adding the shipping route from Supply 1 to Demand 2 will lower (improve) the shipping cost.</u>

From Figure 6.10, we see that the cost to ship from Supply 1 to Demand 2 is $3. The cost to ship along the route in the spanning tree in Figure A.10 is:

$$c_{11} - c_{21} + c_{22} = 10 - 12 + 14 = \$12$$

Since the cost of shipping direct is cheaper, we add the edge which represents the shipping route from Supply 1 to Demand 2. However, we now have a cycle, so we must decide which edge of the cycle is to be removed. To help us decide, we must determine how many items can be rerouted using the newly added shipping route. Figure A.11 shows that the maximum number of items which can be rerouted is 2 and the edge to be removed is the one from Supply 2 to Demand 2. The newly added edge is indicated with a bold line.

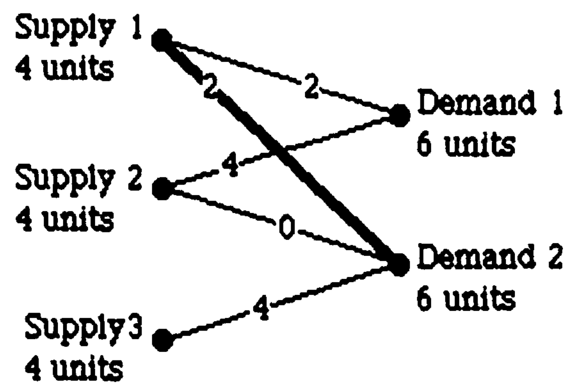


Figure A.11

When this edge is dropped, we have the spanning tree in Figure A.12.

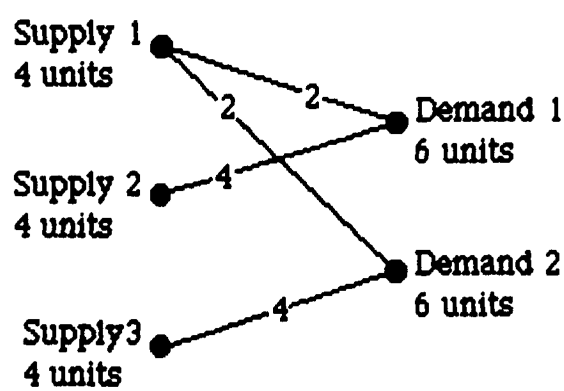


Figure A.12

To determine if we have found the optimal solution, we must repeat the above procedure. That is, we must check all the possibilities for new shipping routes. The only two shipping routes that we could consider adding are from Supply 2 to Demand 2 and from Supply 3 to Demand 1. The shipping cost for one item from Supply 2 to Demand 2 is \$14. The equivalent cost to ship one item in the spanning tree of Figure A.12 is

$$c_{21} - c_{11} + c_{12} = 12 - 10 + 3 = \$5.$$

Since it is cheaper to ship an item from Supply 2 to Demand 2, in the spanning tree of Figure A.12, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Shipping an item from Supply 3 to Demand 1 costs $2. The equivalent cost to ship an item in the spanning tree of Figure A.12 is

$$c_{32} - c_{12} + c_{11} = 8 - 3 + 10 = \$15.$$

Since the cost of shipping direct is cheaper, we add the edge that represents the shipping route from Supply 3 to Demand 1. However, we now have a cycle, so we must decide which edge of the cycle is to be removed. To help us decide, we must determine how many items can be rerouted using the newly added shipping route. Figure A.13 shows that the maximum number of items which can be rerouted is 2 and the edge to be removed is the one from Supply 1 to Demand 1. The newly added edge is indicated with a bold line.

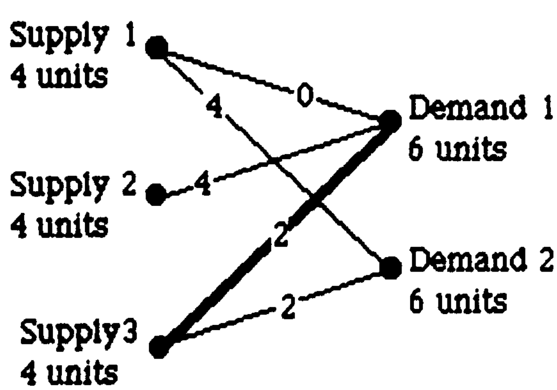


Figure A.13

When this edge is dropped, we have the spanning tree in Figure A.14.

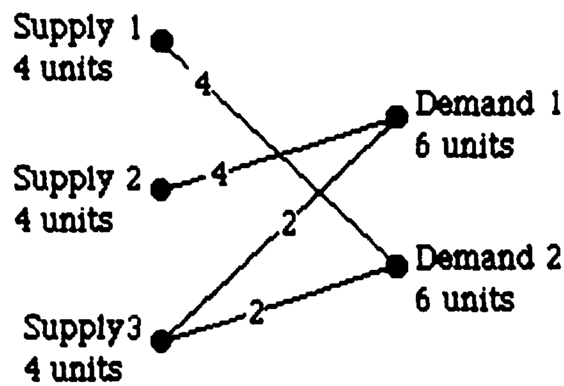


Figure A.14

To determine if we have found the optimal solution, we must repeat the above procedure. That is, we must check all the possibilities for new shipping routes. The only two shipping routes that we need consider adding are from Supply 1 to Demand 1 and from Supply 2 to Demand 2. The shipping cost for one item from Supply 1 to Demand 1 is \$10. The equivalent cost to ship one item in the spanning tree of Figure A.14 is

$$c_{12} - c_{32} + c_{31} = 3 - 8 + 2 = -\$3.$$

Since it is cheaper to ship an item from Supply 1 to Demand 1, in the spanning tree of Figure A.14, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Shipping an item from Supply 2 to Demand 2 costs \$14. The equivalent cost to ship an item in the spanning tree of Figure A.14 is

$$c_{21} - c_{31} + c_{32} = 12 - 2 + 8 = \$18.$$

Since the cost of shipping direct is cheaper, we add the edge that represents the shipping route from Supply 2 to Demand 2. However, we now have a cycle, so we must decide which edge of the cycle is to be removed. To help us decide, we must determine how many items can be rerouted using the newly added shipping route. Figure A.15 shows that the maximum number of items that can be rerouted is 2 and the edge to be removed is the one from Supply 3 to Demand 2. The newly added edge is indicated with a bold line.

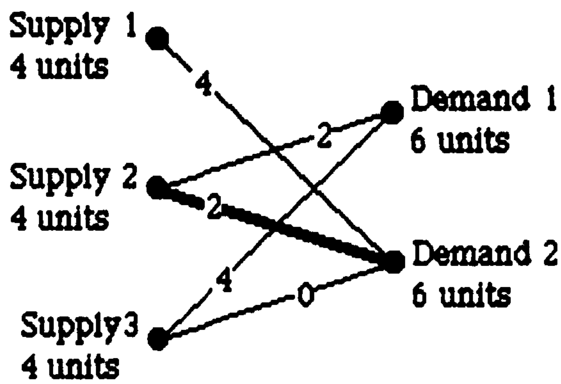


Figure A.15

When this edge is dropped, we have the spanning tree in Figure A.16.

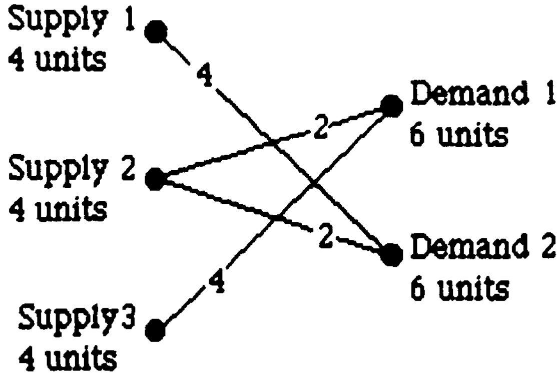


Figure A.16

To determine if we have found the optimal solution, we must repeat the procedure above. That is, we must check all the possibilities for new shipping routes. The only two shipping routes that we could consider adding would be from Supply 1 to Demand 1 and from Supply 3 to Demand 2. The shipping cost for one item from Supply 1 to Demand 1 is \$10. The equivalent cost to ship one item in the spanning tree of Figure A.16 is

$$c_{12} - c_{22} + c_{21} = 3 - 14 + 12 = \$1.$$

Since it is cheaper to ship an item from Supply 1 to Demand 1, in the spanning tree of Figure A.16, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Shipping an item from Supply 3 to Demand 2 costs \$8. The equivalent cost to ship an item in the spanning tree of Figure A.16 is

$$c_{31} - c_{21} + c_{22} = 2 - 12 + 14 = \$4.$$

Since it is cheaper to ship an item from Supply 3 to Demand 2, in the spanning tree of Figure A.16, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Therefore, we have found tne optimal solution and the minimum cost is

$$10x_{11}+ 3x_{12}+ 12x_{21}+ 14x_{22}+ 2x_{31}+ 8x_{32}$$

$$= 10\,(0) + 3\,(4) + 12\,(2) + 14\,(2) + 2\,(4) + 8\,(0)$$

$$= \$72.$$

Determine if adding the shipping route from Supply 3 to Demand 1 will lower (improve) the shipping cost.

From Figure 6.10, we see that the cost to ship from Supply 3 to Demand 1 is \$2. The cost to ship along the route in the spanning tree in Figure A.10 is

$$c_{32} - c_{22} + c_{21} = 8 - 14 + 12 = \$6.$$

Since the cost of shipping direct is cheaper, we add the edge which represents the shipping route from Supply 3 to Demand 1. However, we now have a cycle, so we must decide which edge of the cycle is to be removed. To help us decide, we must determine how many items can be rerouted using the newly added shipping route. Figure A.17 shows that the maximum number of items which can be rerouted is 2 and the edge to be removed is the one from Supply 2 to Demand 1. The newly added edge is indicated with a bold line.

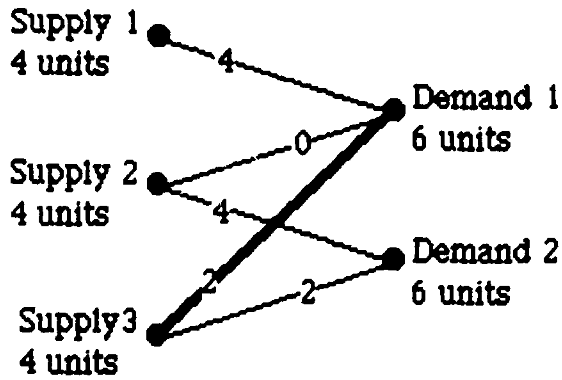


Figure A.17

When this edge is dropped, we have the spanning tree in Figure A.18.

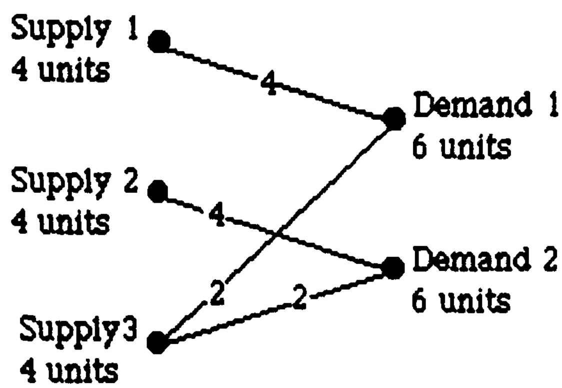


Figure A.18

To determine if we have found the optimal solution, we must repeat the procedure above. That is, we must check all the possibilities for new shipping routes. The only two shipping routes that we could consider adding are from Supply 1 to Demand 2 and from Supply 2 to Demand 1. The shipping cost for one item from Supply 1 to Demand 2 is \$3. The equivalent cost to ship one item in the spanning tree of Figure A.18 is

$$c_{11} - c_{31} + c_{32} = 10 - 2 + 8 = \$16.$$

Since the cost of shipping direct is cheaper, we add the edge which represents the shipping route from Supply 1 to Demand 2. However, we now have a cycle, so we must decide which edge of the cycle is to be removed. To help us decide, we must determine how many items can be rerouted using the newly added shipping route. Figure A.19 shows that the maximum number of items which can be rerouted is 2 and the edge to be removed is the one from Supply 3 to Demand 2. The newly added edge is indicated with a bold line.

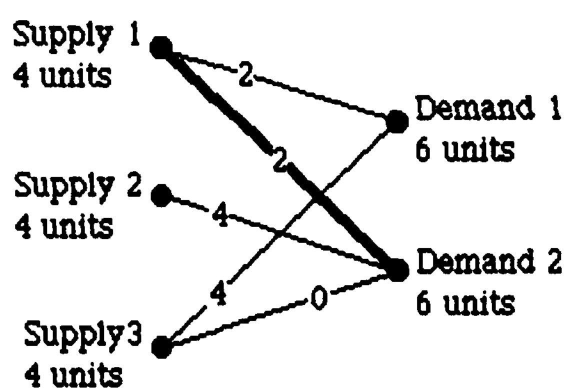


Figure A.19

When this edge is dropped, we have the spanning tree in Figure A.20.

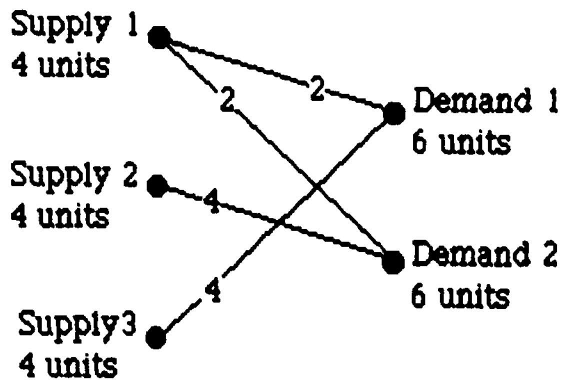


Figure A.20

To determine if we have found the optimal solution, we must repeat the procedure above. That is, we must check all the possibilities for new shipping routes. The only two shipping routes that we could consider adding would be from Supply 2 to Demand 1 and from Supply 3 to Demand 2. The shipping cost for one item from Supply 2 to Demand 1 is \$12. The equivalent cost to ship one item in the spanning tree of Figure A.20 is

$$c_{23} - c_{13} + c_{11} = 14 - 3 + 10 = \$21.$$

Since the cost of shipping direct is cheaper, we add the edge that represents the shipping route from Supply 2 to Demand 1. However, we now have a cycle, so we must decide which edge of the cycle is to be removed. To help us decide, we must determine how many items can be rerouted using the newly added shipping route. Figure A.21 shows that the maximum number of items which can be rerouted is 2 and the edge to be removed is the one from Supply 3 to Demand 2. The newly added edge is indicated with a bold line.

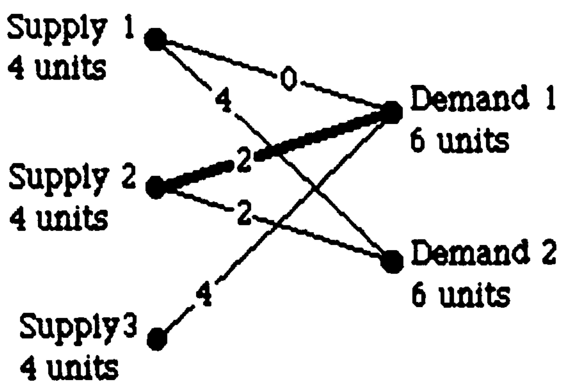


Figure A.21

When this edge is dropped, we have the spanning tree in Figure A.22.

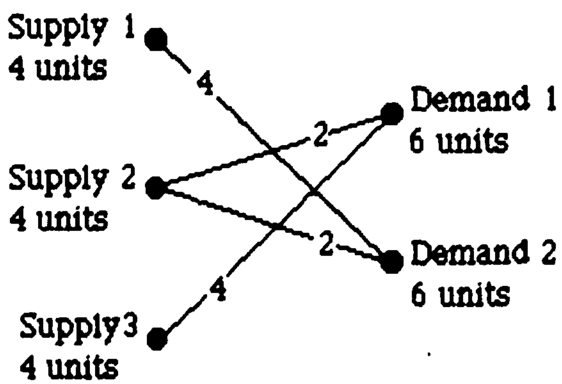


Figure A.22

To determine if v ↩ have found the optimal solution, we must repeat the procedure above. That is, we must check all the possibilities for new shipping routes. The only two shipping routes that we need consider adding are from Supply 1 to Demand 1 and from Supply 3 to Demand 2. The shipping cost for one item from Supply 1 to Demand 1 is $10. The equivalent cost to ship one item in the spanning tree of Figure A.22 is

$$c_{12} - c_{22} + c_{21} = 3 - 14 + 12 = \$1.$$

Since it is cheaper to ship an item from Supply 1 to Demand 1 in the spanning tree of Figure A.22 than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Shipping an item from Supply 3 to Demand 2 costs \$8. The equivalent cost to ship an item in the spanning tree of Figure A.22 is

$$c_{31} - c_{21} + c_{22} = 2 - 12 + 14 = \$4.$$

Since it is cheaper to ship an item from Supply 1 to Demand 1, in the spanning tree of Figure A.22, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Therefore we have found the optimal solution and the minimum cost is

$$10x_{11} + 3x_{12} + 12x_{21} + 14x_{22} + 2x_{31} + 8x_{32}$$

$$= 10\,(0) + 3\,(4) + 12\,(2) + 14\,(2) + 2\,(4) + 8\,(0)$$

$$= \$72.$$

Notice that Figures A.16 and A.22 are identical. Thus, it did not matter which edge we considered adding first since both give the same optimal solution.

<u>Exercise 6.3</u>

Figure A.23 illustrates the Mr. Potatohead transportation problem.

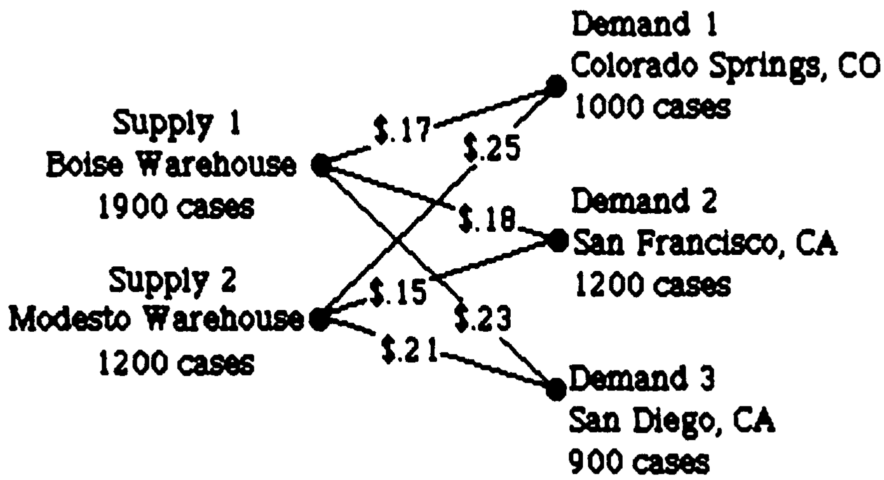


Figure A.23

Let x_{ij} be the amount shipped from supply i to demand j, where $i=1, 2$ and $j=1, 2, 3$. The transportation problem written in standard format is:

Minimize Cost:

$$.17x_{11}+.18x_{12}+.23x_{13}+.25x_{21}+.15x_{22}+.21x_{23}$$

Subject to:

$$
\begin{array}{llllllll}
x_{11} & +x_{12} & +x_{13} & & & & =1900 & \text{(Supply 1)} \\
 & & & x_{21} & +x_{22} & +x_{23} & =1200 & \text{(Supply 2)} \\
x_{11} & & & +x_{21} & & & =1000 & \text{(Demand 1)} \\
 & x_{12} & & & +x_{22} & & =1200 & \text{(Demand 2)} \\
 & & x_{13} & & & +x_{23} & =900 & \text{(Demand 3)}
\end{array}
$$

and all $x_{ij} \geq 0$.

Using the Northwest Rule, we get the spanning tree in Figure A.24.

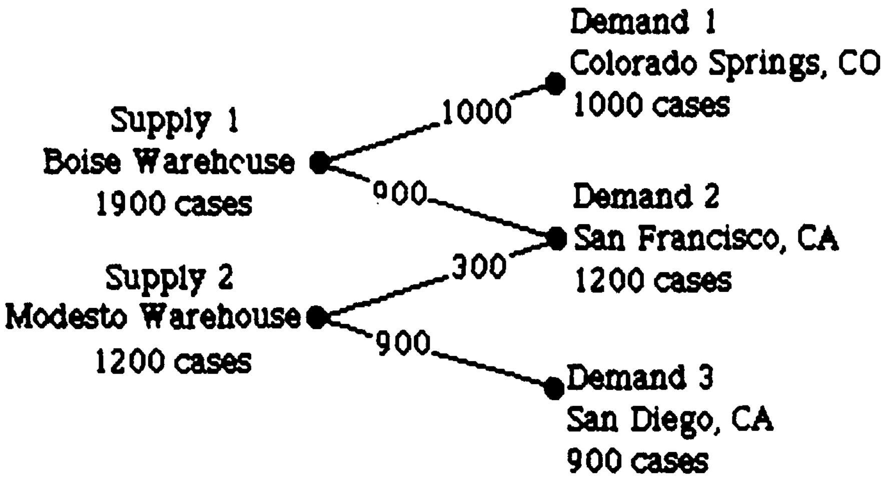


Figure A.24

The cost that we will try to improve (minimize) is:

$$.17x_{11} + .18x_{12} + .23x_{13} + .25x_{21} + .15x_{22} + .21x_{23}$$

$$= .17(1000) + .18(900) + .23(0) + .25(0) + .15(300) + .21(900)$$

$$= \$566.$$

The two shipping routes that we need to check to determine if either should be added to the spanning tree are from Boise to San Diego and from Modesto to Colorado Springs.

Solution 1: Determine if adding the shipping route from Boise to San Diego will lower (improve) the shipping cost.

From Figure A.23, we see that the cost of shipping a case of chips from Boise to San Diego is $.23. The cost to ship along the route in the spanning tree in Figure A.24 is

$$c_{11} - c_{22} + c_{23} = .18 - .15 + .21 = \$.24.$$

Since the cost of shipping direct is cheaper, we add the edge which represents the shipping route from Boise to San Diego. However, we now have a cycle, so we must decide which edge of the cycle is to be removed. To help us decide, we must determine how many items can be rerouted using the newly added shipping route. Figure A.25 shows that the maximum number of items which can be rerouted is 900. Since we have two edges that become zero, either edge can be removed. The newly added edge is indicated with a bold line.

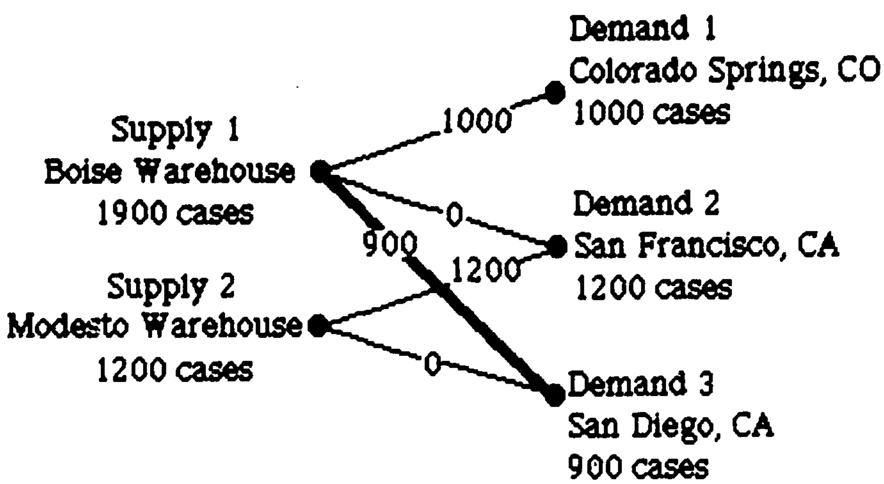


Figure A.25

We can drop only one of these edges, otherwise we would not end up with a spanning tree. Figures A.26 and A.27 show the two resulting spanning trees.

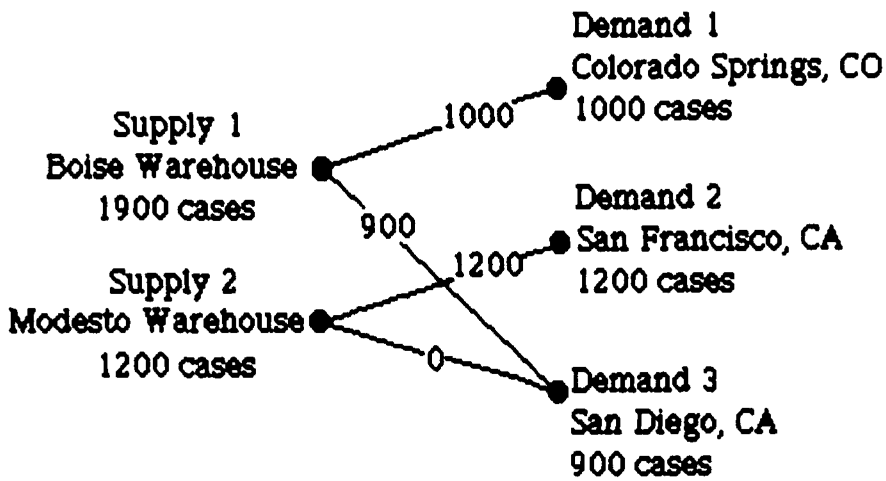


Figure A.26

To determine if the spanning tree in Figure A.26 will give us the optimal solution, we must check all the possibilities for new shipping routes. The only two shipping routes that we need consider adding are from Boise to San Francisco and from Modesto to Colorado Springs. The shipping cost for one item from Boise to San Francisco is \$.18. The equivalent cost to ship one item in the spanning tree of Figure A.26 is

$$c_{13} - c_{23} + c_{22} = .23 - .21 + .15 = \$.17.$$

Since it is cheaper to ship an item from Boise to San Francisco in the spanning tree of Figure A.26 than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Shipping an item from Modesto to Colorado Springs costs \$.25. The equivalent cost to ship an item in the spanning tree of Figure A.26 is

$$c_{22} - c_{12} + c_{11} = .15 - .18 + .17 = \$.14.$$

Since it is cheaper to ship an item from Modesto to Colorado Springs, in the spanning tree of Figure A.26, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Therefore we have found the optimal solution and the minimum cost is

$$.17x_{11} + .18x_{12} + .23x_{13} + .25x_{21} + .15x_{22} + .21x_{23}$$

$$= .17(1000) + .18(0) + .23(900) + .25(0) + .15(1200) + .21(0)$$

$$= \$557.$$

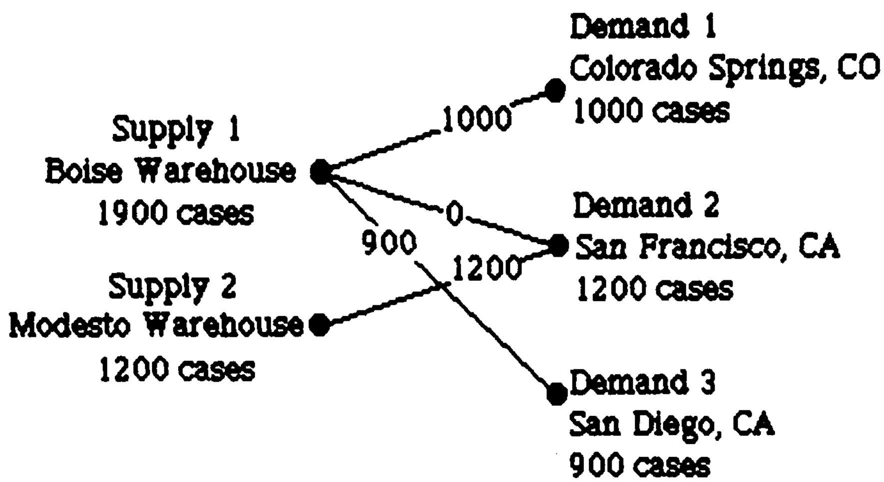


Figure A.27

We now show that the spanning tree in Figure A.27 will also give us the optimal solution. We begin by checking all the possibilities for new shipping routes. The only two shipping routes that we need consider adding are from Modesto to Colorado Springs and from Modesto to San Diego. The shipping cost for one item from Modesto to Colorado Springs is \$.25. The equivalent cost to ship one item in the spanning tree of Figure A.27 is

$$c_{22} - c_{12} + c_{11} = .15 - .18 + .17 = \$.14.$$

Since it is cheaper to ship an item from Modesto to Colorado Springs in the spanning tree of Figure A.27 than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Shipping an item from Modesto to San Diego costs \$.21. The equivalent cost to ship an item in the spanning tree of Figure A.27 is

$$c_{22} - c_{12} + c_{13} = .15 - .18 + .23 = \$.20.$$

Since it is cheaper to ship an item from Modesto to Colorado Springs, in the spanning tree of Figure A.26, than it is to ship it directly, we decide not to add this shipping route to the spanning tree. Therefore we have found the optimal solution and the minimum cost is

$$.17x_{11} + .18x_{12} + .23x_{13} + .25x_{21} + .15x_{22} + .21x_{23}$$

$$= .17(1000) + .18(0) + .23(900) + .25(0) + .15(1200) + .21(0)$$

$$= \$557.$$

Therefore, both the spanning trees in Figures A.26 and A.27 give us the same optimal solution of \$557.

Solution 2: Determine if adding the shipping route from Modesto to Colorado Springs will lower (improve) the shipping cost.

From Figure A.23 we see that the cost of shipping a case of chips from Modesto to Colorado Springs is \$.25. The equivalent cost using the spanning tree in Figure A.24 is

$$c_{22} - c_{12} + c_{11} = .15 - .18 + .17 = \$.14.$$

Since the cost is greater if we were to ship directly from Modesto to Colorado Springs, we will not consider adding this route. We must now check adding the shipping route from Boise to San Diego, but this is Solution 1 above.

APPENDIX C

Application III

Contents of Application III

Application III

Linear Algebra Applied to Physics

Determining Small Vibrations in Conservative Elastic Systems

Linear Algebra Prerequisites: Being able use eigenvalues and eigenvector to diagonalize a symmetric matrix.

Prerequisite Knowledge in Physics: None.

Other Prerequisite Knowledge: A background in solving basic differential equations would be helpful. However, Appendix A contains this basic information.

Section 1 Introduction

In this study we will look at small vibrations. In particular, the small vibrations which we will study are in a system with an **equilibrium configuration** which is a position where the system remains at rest. An example of a system in its equilibrium configuration is the simple pendulum as seen in Figure 1.1. The simple pendulum consists of a ball attached to a taut wire, anchored above, which can swing in the vertical plane. The weight of the wire is negligible compared to the weight of the ball.

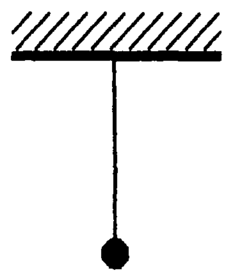


Figure 1.1

We say that a system has a **stable equilibrium configuration** if after a small displacement, the system tends to return to its equilibrium configuration. There are different types of equilibrium depending on the nature of the system. We are interested in the type of equilibrium found in an **elastic system**. This is a system which has the following two characteristics: 1) the system has a stable equilibrium configuration and 2) a small displacement from equilibrium creates forces which tend to restore the system to its stable equilibrium configuration. A displacement from equilibrium is called **strain** and the force which restores the system to equilibrium is called **stress**. Thus, stress is a function of strain. The simple pendulum in Figure 1.1 is also an example of an elastic system in its stable equilibrium configuration.

The total energy in an elastic system is composed of two types of energy, kinetic and potential. We will begin by considering the intuitive definitions of these terms and then discuss their formulas. **Kinetic energy** is the energy a body possesses because it is in motion. Before we can write the formula for kinetic energy, we must be able to describe the system mathematically. In any system there is a minimum number of coordinates that are required to fully describe the configuration of the system. In general, the number of coordinates is equal to the number of "particles" in the system times the dimension of the system. In the case of the simple pendulum, the ball is the only particle in the system. The dimension of the system is one, because the position of the ball can be described using the angle made by the pendulum compared to the position of the pendulum in its equilibrium configuration, as seen in Figure 1.1. Therefore, the number of coordinates needed to describe the simple pendulum is one. The velocity of the system can also be written in terms of the coordinates which describe the configuration of the system. To be able to do this, we must specialize our notation. If n coordinates $(x_1, x_2, \ldots, x_n)$ are required to describe the system, then each x_i represents a Cartesian coordinate of one of the particles in the system. For example, if we have two particles moving in the xy-plane, which has dimension two, we will need four coordinates to describe the system. The four coordinates (x_1, x_2, x_3, x_4) represent the

Cartesian coordinates of the particles in the system; that is, x_1 and x_2 represent x- and y-coordinates of the first particle, and x_3 and x_4 represent the x- and y-coordinates of the second particle. From this we see that the velocity of the system can be expressed in terms of the velocity of each coordinate. The velocity vector for a system with n coordinates can be written in terms of its velocity components

$$\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right).$$

The kinetic energy of the system is equal to the sum of one half the square of each velocity component times the mass of the particle which the coordinate describes. If we let T represent kinetic energy and m_i the mass of the particle which is described using the Cartesian coordinate x_i, then our formula becomes

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{dx_i}{dt}\right)^2.$$

To have a **conservative system**, there must exist a function whose partial derivative with respect to any coordinate, say x_i, is equal to the negative value of the force in the direction represented by that coordinate. This function is called the **potential energy function**. We can describe the relationship between this function and the forces in the system by the equation

$$\frac{\partial}{\partial x_i}(\text{potential energy function}) = -(\text{force in the } x_i \text{ direction})$$

From now on, we will assume that we are always in a conservative system. In addition, if the potential energy function is not time dependent $\left(\frac{\partial}{\partial t}(\text{potential energy function}) = 0\right)$, then in our conservative elastic system the total energy of the system is constant and is the sum of the kinetic and potential energies. Also, when the strain of the system is zero (the system is in its equilibrium configuration, so $x_i = 0$ for all i), then the partial derivative of the potential energy function with respect to any variable must equal zero. This statement can be interpreted in the following two ways: 1) in the equilibrium configuration the potential energy function is at a minimum and 2) the restoring forces are equal in magnitude and of opposite sign to the forces that created the displacement. This statement also tells us that the potential energy function can not contain linear terms which have nonzero constant coefficients in any of the x_i. To see why this is true, let us assume the potential energy function contains a nonzero linear term cx_i (c is a nonzero constant). Then take the partial derivative of it with respect to x_i. Setting x_i equal to zero, we find the nonzero constant c is equal to zero, which is a contradiction. Therefore, we conclude that c must be zero and the potential energy function does not contain a nonzero linear term cx_i. Also, it does not matter if the potential energy function has a constant term or not, because when we differentiate the function with respect to any x_i ($i=1, 2, \ldots, n$), the constant becomes zero. Thus, if we write the potential energy function in its Taylor series expansion, the non-constant part starts with quadratic and terms of higher powers (which may also contain a constant term). When we differentiate the potential energy function with respect to x_i (for $i=1, 2, \ldots, n$), we obtain a linear combination of the variables $x_1, x_2, \ldots, x_n$ plus higher order or mixed terms (for example $x_1 x_n$ or $x_1 x_2 x_n$). If we ignore the higher order terms, then the linear part which remains gives us the specific relationship of stress to strain, which is known as Hooke's Law. In general, Hooke's Law states that "stress is a linear transformation operating on strain." Intuitively, we would say, restoring forces are linearly proportional to the displacement of the mass from equilibrium. If the non-constant potential energy function starts with a power greater than two, it is possible to use an approximation to find the relationship between stress and strain in which the still higher power terms in the partial derivative of the potential energy function have been ignored However, this is no longer a linear function.

We will consider two approaches to the formulation of a differential equation which models a system. The first approach is developed using Newton's second and third laws of motion, which are stated below for convenience.

Second Law	The mass of the body times the acceleration of the body is equal to the force acting on the body.
Third Law	For every action there is an opposite and equal reaction.

From these laws we derive the differential equation which models a conservative elastic system

$$\text{mass} \times \frac{d^2(\text{displacment})}{dt^2} = \text{restoring force}$$

or

$$\text{mass} \times \frac{d^2(\text{strain})}{dt^2} = \text{stress}. \tag{1.1}$$

In the second approach, instead of using the direct application of Newton's laws, we will consider a method developed by Joseph Lagrange, a French mathematician. This very elegant and sophisticated method can be applied to systems which are more general than the ones we are considering here. Since our system is conservative and elastic, the energy is constant and equal to the sum of the kinetic and potential energies. If we let V represent potential energy and E represent the total energy in an n coordinate system, then we have

$$E = T + V = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{dx_i}{dt}\right)^2 + V$$

To make it easier to express this differential equation, we will introduce a type of notation you may not have used before. The derivative $\frac{dx}{dt}$ will be written as $\dot{x}$, where the single dot above the variable x indicates that one derivative of x, with respect to time, has been taken. This idea can be extended so that $\ddot{x}$ indicates that two derivatives of x with respect to time have been taken. This notation is used to rewrite the equation for total energy.

(1.2) $$E = T + V = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{x}_i^2 + V$$

We wish to derive the equations of motion which can be used to model conservative elastic systems. Our first step is to find the partial derivative of Equation (1.2) with respect to each of the coordinates. Since the procedure is the same when taking the partial derivative with respect to each coordinate, we will only find the partial derivative of the function for total energy (a constant) with respect to the x_i th coordinate. We obtain

$$0 = \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i}$$

or

(1.3) $$-\frac{\partial V}{\partial x_i} = \frac{\partial T}{\partial x_i}.$$

Since the mass of each particle is known, the partial derivative of the kinetic energy with respect to the coordinate x_i is

$$\frac{\partial T}{\partial x_i} = \frac{1}{2}m_i\left[2\dot{x}_i\frac{\partial \dot{x}_i}{\partial x_i}\right] = m_i\frac{\partial x_i}{\partial t}\frac{\partial \dot{x}_i}{\partial x_i} = m_i\frac{\partial \dot{x}_i}{\partial t} = m_i\ddot{x}_i.$$

Substituting this into Equation (1.3), we obtain the restoring force of the x_i coordinate.

(1.4) $$-\frac{\partial V}{\partial x_i} = m_i\ddot{x}_i.$$

Since the kinetic energy is expressed in terms of $\dot{x}_i$, we can find $\frac{\partial T}{\partial \dot{x}_i}$.

$$\frac{\partial T}{\partial \dot{x}_i} = \frac{1}{2}m_i\left[2\dot{x}_i\right] = m_i\dot{x}_i$$

We now differentiate this equation with respect to time to obtain

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_i}\right] = m_i\frac{d\dot{x}_i}{dt} = m_i\ddot{x}_i.$$

We see that the right side of this equation is the same as the right side of Equation (1.4). Equating the two, we obtain the equation of motion for the x_i th coordinate.

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_i}\right] = -\frac{\partial V}{\partial x_i}$$

Therefore, the equations of motion which model our conservative elastic system with n coordinates are

$$\text{(1.5)} \qquad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = -\frac{\partial V}{\partial x_1},\ \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = -\frac{\partial V}{\partial x_2},\ \ldots,\ \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_n}\right] = -\frac{\partial V}{\partial x_n}.$$

We will model the simple pendulum of Figure 1.1 using both the method which applies Newton's laws directly and the equations of motion formulated by Lagrange. Since we are only interested in small vibrations of a system, let us discuss the conditions under which the vibrations in the pendulum system remain small. In a conservative elastic system the total energy of the system is constant and is the sum of the kinetic and potential energies. The potential energy of the pendulum system is determined by the displacement of the ball from its equilibrium position. Imagine the pendulum in Figure 1.1 being placed very close to its equilibrium position and released as in Figure 1.2.

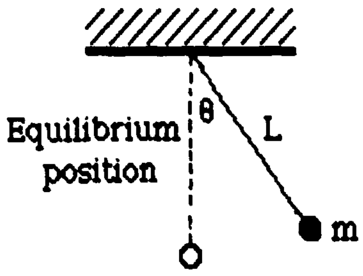


Figure 1.2

Since the potential energy is small to start with (the displacement from equilibrium is small) and we are in a conservative system, we know that it will remain small. Because the displacement stays small, the angle θ will always be small. Thus, the vibrations of this system can only be small vibrations.

Let us model the simple pendulum system using the method which applies Newton's laws directly. To keep this example simple we will only consider the positive region which is to the right of the equilibrium configuration in Figure 1.3(a). Let L represent the length of the pendulum, m be the mass of the ball at the end of the pendulum, θ the angle the pendulum makes with respect to the equilibrium configuration, and s represent the length of the arc the ball travels. The force pulling the ball down is the mass of the ball times the gravitational constant g.

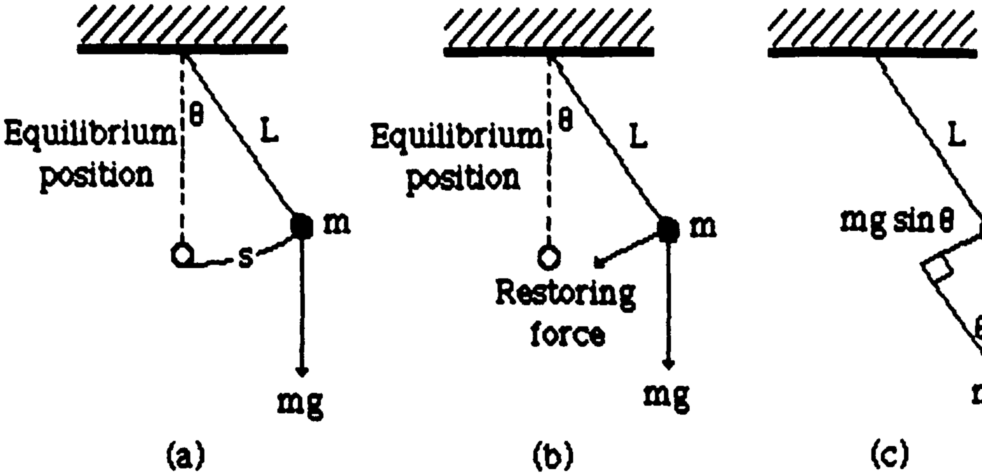


Figure 1.3

We will apply Newton's laws of motion to model the pendulum system by using Equation (1.1). Thus, we need to determine the stress and the strain of the system. Since stress or restoring force is the force trying to return the ball to its equilibrium configuration, we must resolve the force on the ball (mg) into its component forces. Figure 1.3(b) shows the restoring force is the component of force on the ball along the arc length. Since $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, the magnitude of the restoring force is $mg\sin\theta$ as seen in Figure 1.3(c). We will need a minus sign to indicate that the restoring force is opposite in direction to the force which originally moved the ball from its equilibrium configuration. Thus, the restoring force or stress is equal to $-mg\sin\theta$. The strain is the displacement of the ball from the equilibrium position. This distance is the arc length s, which can also be described using the equation $s=L\theta$. Substituting these values for stress and strain into equation (1.1) gives

$$m \frac{d^2}{dt^2}\left[L\,\theta\right] = -\,mg\,\sin\theta$$

Taking the second derivative of $L\theta$ with respect to time, this equation becomes

$$m\,L\ddot{\theta} = -\,mg\,\sin\theta$$

Simplifying and moving all terms to the left side of the equation, we get the second order differential equation that models our conservative elastic system.

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

We now model the simple pendulum system using the equations of motion formulated by Lagrange. Since θ is the only coordinate needed to describe the system, we will only need to use one of the equations of motion found in Equation (1.5).

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] = -\,\frac{\partial V}{\partial \theta}$$

Thus, we need to find both the kinetic energy and potential energy of the system. The kinetic energy is one half the mass of the ball times the square of the velocity. The velocity is the first derivative of the distance with respect to time.

$$\text{velocity} = \frac{d}{dt}[\,\text{distance}\,] = \frac{d}{dt}\left[L\theta\right] = L\,\frac{d\theta}{dt} = L\dot{\theta}$$

Therefore, the equation describing the kinetic energy becomes

$$T = \frac{1}{2}\,m\,[L\,\dot{\theta}\,]^2 = \frac{1}{2}\,m\,L^2\,\dot{\theta}^2$$

Since potential energy is the energy needed to restore the system to equilibrium, it is equal to weight of the ball (mass of the ball times the gravitational constant g) times the height of the ball above the reference point. Since the ball is below the reference point, V is negative. Using the fact that $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, we determine the distance of the ball below the reference point to be $L\cos\theta$ as seen in Figure 1.4.

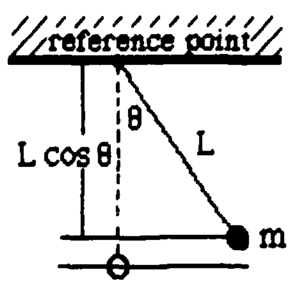


Figure 1.4

Thus, the potential energy is

$$V = -mgL\cos\theta\,.$$

Now, we substitute the appropriate partial derivatives of the kinetic and potential energy equations into the equation of motion. First, we find the left side of the equation of motion by differentiating the kinetic energy equation with respect to $\dot{\theta}$.

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} m L^2 (2\dot{\theta}) = m L^2 \dot{\theta}$$

Then differentiating with respect to time, we obtain

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] = m L^2 \ddot{\theta}$$

Second, the right side of the equation of motion is

$$-\frac{\partial V}{\partial \theta} = -\left[-m g L \frac{\partial(\cos\theta)}{\partial \theta}\right] = -[-m g L(-\sin\theta)] = -m g L\sin\theta$$

Equating the two sides, the equation of motion becomes

$$m L^2 \ddot{\theta} = -m g L\sin\theta$$

Simplifying and moving all terms to the left side of the equation, we obtain the second order differential equation that models our conservative elastic system. As expected, this is the same equation which we found by applying Newton's laws.

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{1.6}$$

Let us pause for a moment and discuss the relationship between the potential energy function and the component of force tangent to the path the ball travels (that is, in the direction of arc length). Recall that in a conservative system, the partial derivative of the potential energy function with respect to any direction, gives the negative of the force in that direction. That is, $\frac{\partial V}{\partial s} = -F_s$ where F_s is the force in the direction of the arc length s. First, we need to write the potential energy function in terms of arc length s. We will use the fact that $s=L\theta$.

$$V = -mgL\cos\left(\frac{s}{L}\right)$$

We continue by differentiating this with respect to s to obtain

$$\frac{\partial V}{\partial s} = -m\,g\,L\left[-\frac{1}{L}\sin\left(\frac{s}{L}\right)\right] = m\,g\sin\theta$$

Thus, $F_s = -\frac{\partial V}{\partial s} = -mg\sin\theta$ is the restoring force, since the simple pendulum is described using only one coordinate. (Recall the discussion following Figure 1.3.)

So far we have found the second order differential equation which models the simple pendulum system using two different methods. Now, we are ready to consider how Equation (1.6) can be solved. Since this equation involves $\sin\theta$, we know it is a nonlinear differential equation. (See

Appendix A for definition.) One technique used to find the exact solution (if that is possible) of a second order nonlinear differential equation is to first reduce it to a first order differential equation. Recall that, the equation of motion was derived from Equation (1.7). Since Equation (1.6) was found by using the equation of motion, we can use Equation (1.7) as our first order differential equation.

$$E = T + V = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt}\right)^2 - m g L \cos\theta \tag{1.7}$$

We begin by solving for the squared derivative $\left(\frac{d\theta}{dt}\right)^2$ in Equation (1.7).

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2(E + m g L \cos\theta)}{m L^2} \tag{1.8}$$

Let us pause for a moment to assure ourselves that we could legitimately use Equation (1.7) as our first order differential equation. We will differentiate Equation (1.8) with respect to time and see that we get Equation (1.6).

$$2\,\dot{\theta}\,\ddot{\theta} = \frac{2}{m L^2}\left[0 + mgL(-\dot{\theta}\sin\theta)\right]$$

When simplified, we see this is Equation (1.6).

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

Taking the square root of both sides of Equation (1.8) and recalling that we are only considering the positive region to the right of the equilibrium configuration, we obtain

$$\frac{d\theta}{dt} = \sqrt{\frac{2(E + m\,g\,L\cos\theta)}{mL^2}}\ .$$

One way to solve the differential equation above for t, is to isolate dt on one side of the equation and dθ on the other. This technique is sometimes called separation of variables.

$$dt = \sqrt{\frac{mL^2}{2(E + m\,g\,L\cos\theta)}}\ d\theta$$

Integrating both sides of the above equation, we obtain

$$t = \int \sqrt{\frac{mL^2}{2(E + m\,g\,L\cos\theta)}}\ d\theta\ .$$

This is an elliptic integral which can not be expressed in terms of elementary functions. Thus, to get any information about the solution to Equation (1.6), we must resort to numerical approximations or use the fact that we are dealing with a system involving small vibrations. We also note that the question of finding the inverse function $\theta = \theta(t)$ of the function above is, at best, a numerical approximation problem and is not even useful in predicting values of θ at a given time t, since we are dealing with small vibrations. As a point of interest, if we were not considering small vibrations, then the function $t = t(\theta)$ and its inverse function $\theta = \theta(t)$ would be the only tools with which we could obtain information about the system.

Using the fact that we are dealing only with small vibrations, we consider the factor $\sin\theta$, which makes Equation (1.6) a nonlinear differential equation. We can write $\sin\theta$ as a Taylor series expanded about zero.

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,\theta^{2n+1} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

Since we are considering only small values of θ, the terms in the expansion above which contain powers of θ are, in practice, ignored (a very small number raised to a power greater than one becomes even smaller). Any time terms are ignored we expect a certain amount of error. To determine the exact amount of error would require the same type of calculation that it would take to solve the original equation. However, since we are considering only small vibrations, we are assured the amount of error will not affect the resulting solution. Thus, using the Taylor series expansion for $\sin\theta$ we see that $\sin\theta$ can be replaced by θ, for small values of θ. This substitution is only valid when we are dealing with small vibrations. Using this substitution, the second order differential equation (Equation (1.6)) becomes

$$\ddot{\theta} + \frac{g}{L}\,\theta = 0 .$$

This is a second order linear differential equation whose solution is found using basic techniques from differential equations. (Basic solution techniques are found in Appendix A.) We obtain

$$\theta = c_1 \cos\left(\sqrt{\frac{g}{L}}\; t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}\; t\right).$$

In summary, we have investigated two different ways to model a conservative elastic system. One method applies Newton's second and third laws directly to the system to create the differential equation. The other method uses a technique developed by Lagrange, which was much easier to generalize and could be applied to many different types of systems. The derived equations of motion, greatly simplify the amount of work necessary to model a conservative elastic system. From these techniques, we found the second order nonlinear differential equation that models the simple pendulum. Since we considered only small vibrations, we found that the equation could be represented by a second order linear differential equation which has an elementary solution.

Section 2 Linear Spring-Weight Systems

In the last section we considered a system which only needed one coordinate to completely describe the system. We now look at higher dimensional systems, such as spring-weight systems in which more than one coordinate is required to specify the state of the system. A spring-weight system is a conservative elastic system with a stable equilibrium position occurring when all of the coordinates are set equal to zero. To become familiar with spring-weight systems, we will first consider the one dimensional case. Figure 2.1 shows the system in its equilibrium configuration (the spring is not being stretched or compressed) where m is the mass of the block and L is the natural length of the spring. We are considering the spring-weight system moving along a horizontal track rather than hanging vertically so that we do not have the added complication of describing how gravity affects the system.

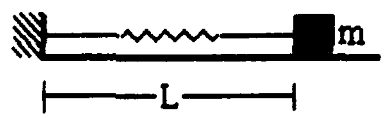


Figure 2.1

To determine the number of coordinates we need to describe this system, recall the formula given in Section 1. (The number of coordinates = (number of particles) times (dimension of the system).) The only particle in the system is the block and since the block is moving along a horizontal track, the dimension of the system is one. Thus, we need only one coordinate x, to describe the system. Imagine the block being moved to the right causing the spring to be stretched x units. This is shown in Figure 2.2.

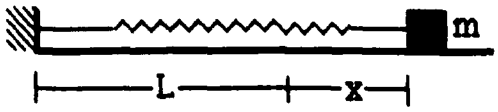


Figure 2.2

To describe the energy of this system we again need to find both the kinetic energy and potential energy. The kinetic energy is one half the mass of the block times the velocity squared. The equation describing the kinetic energy is

$$T = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = \frac{1}{2} m \dot{x}^2 .$$

The energy stored in the spring or the potential energy of the spring is one half the spring constant times the square of the distance that the spring is stretched. From the laws of physics we know that the external force acting on the spring is proportional to the increase in length of the spring. We call the constant of proportionality that allows us to write this relationship as an equation, the spring constant or the stiffness of the spring and each spring has its own specific spring constant. If we let k represent the spring constant and x the displacement of the spring from equilibrium, then the equation for potential energy is

$$V = \frac{1}{2} k x^2 .$$

Since x is the only coordinate needed to describe the system, we will only need to use one of the equations of motion found in Equation (1.5).

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] = -\frac{\partial V}{\partial x}$$

To find the left side, we first differentiate the kinetic energy function with respect to $\dot{x}$.

$$\frac{\partial T}{\partial \dot{x}} = m\,\dot{x}$$

Now differentiate this equation with respect to time.

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}}\right] = m\,\ddot{x}$$

To find the right side of the equation of motion, we differentiate the potential energy function with respect to x.

$$-\frac{\partial V}{\partial x} = -k\,x$$

Equating these two, the equation of motion becomes

$$m\ddot{x} = -k\,x.$$

Simplifying and rearranging terms, the differential equation which models this system is

$$\ddot{x} + \frac{k}{m}\,x = 0.$$

This is a second order linear differential equation whose solution is found using basic techniques from differential equations. (See Appendix A.) Note, the similarity between this differential equation and the one that models the simple pendulum.

$$x = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

To be able to model higher dimensional spring-weight systems, we need to study the theory which describes the energy of the system in general terms. In the one dimensional spring-weight system there was only one coordinate which we labeled as x and it was expressed in terms of time. The kinetic energy of the system was described using the first derivative of this coordinate with respect to time, while the potential energy was expressed in terms of the coordinate. If we are working with a higher dimensional system which has n coordinates, say $x_1, x_2, \ldots, x_n$, then the kinetic energy will be described using the first derivative with respect to time of each of the coordinates and the potential energy will be expressed in terms of these n coordinates. In general we have

Kinetic Energy $\qquad T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\frac{dx_i}{dt}\right)^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{x}_i^{\,2}$

Potential Energy $\qquad V = \sum_{i=1}^{s} V_i \quad$ where V_i is the potential energy of each spring and s is the number of springs.

Since stable equilibrium occurs when $x_1 = x_2 = \ldots = x_n = 0$, we may assume that the energy of the system is at a minimum in stable equilibrium. This means the derivative with respect to any variable must be zero when that variable equals zero. Thus, if we have a function which we wish to expand using its Taylor's series expansion, as we did with $\sin\theta$ in Section 1, the expansion can not have a nonzero linear term. For if it did and we took the derivative of it, we would end up with a nonzero constant. Subsequently, when all variables are set equal to zero, the constant would remain, indicating that we do not have stable equilibrium, a contradiction. Therefore, the Taylor series expansion for the potential energy does not have linear terms. However, this expansion may have constant terms.

Let us return to the spring-weight system. Figure 2.3 shows a system in equilibrium with two blocks having the same mass m and three springs possessing the same length and spring constant. To determine the number of coordinates needed to describe this system, we need to recall the formula given in Section 1. (The number of coordinates = (number of particles) times (dimension of system).) The two particles in the system are the two blocks and since both blocks are moving along a horizontal track, the dimension of the system is one. Thus, we will need two coordinates, x_1 and x_2, to describe the system.

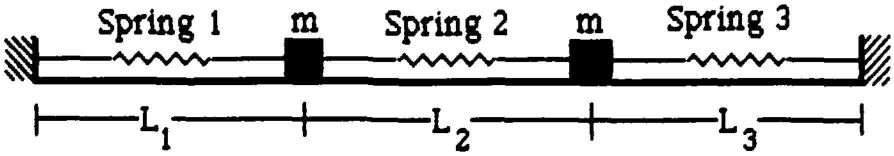


Figure 2.3

Imagine the two masses are moved to the right causing the first two springs to stretch by different amounts and causing the third spring to be compressed. This is depicted in Figure 2.4.

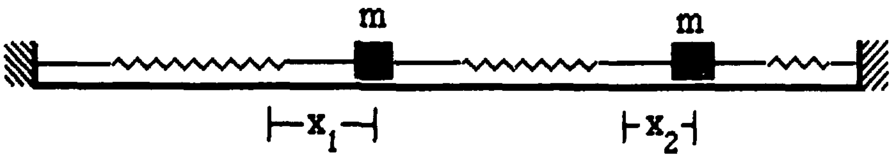


Figure 2.4

Now we determine the second order differential equation that models this system. Thus, we need to find the kinetic energy and the potential energy of the system. The equation below describes the kinetic energy of the system shown in Figure 2.4.

$$T = \frac{1}{2}\left[m\left(\frac{dx_1}{dt}\right)^2 + m\left(\frac{dx_2}{dt}\right)^2\right] = \frac{1}{2}m\left[\dot{x}_1^2 + \dot{x}_2^2\right]$$

The potential energy of the system is the sum of the potential energies of each spring. Spring 1 is stretched from its equilibrium position by the amount x_1, so the potential energy for spring 1 is $V_1 = \frac{1}{2}k\,x_1^2$. Spring 2 is stretched from its equilibrium position by the amount $x_2 - x_1$, so that

$V_2 = \frac{1}{2}k\left(x_2 - x_1\right)^2$ is the potential energy for spring 2. Spring 3 is compressed from its equilibrium position by the amount x_2. Thus, the potential energy for spring 3 is $V_3 = \frac{1}{2}k\,x_2^2$. Therefore, the potential energy of the system is

$$V = \sum_{i=1}^{3} V_i = \frac{1}{2}k\left[x_1^2 + \left(x_2 - x_1\right)^2 + x_2^2\right] = \frac{1}{2}k\left[2x_1^2 - 2x_1x_2 + 2x_2^2\right].$$

Since this system is described using two variables, x_1 and x_2, our two equations of motion are

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = -\frac{\partial V}{\partial x_1} \quad \text{and} \quad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = -\frac{\partial V}{\partial x_2}.$$

First, we determine the left side of each equation of motion by differentiating the kinetic energy with respect to $\dot{x}_1$ and $\dot{x}_2$.

$$\frac{\partial T}{\partial \dot{x}_1} = \frac{1}{2}\, m\left[2\dot{x}_1 + 0\right] = m\dot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{2}\, m\left[0 + 2\dot{x}_2\right] = m\dot{x}_2$$

Then, differentiating each of these equations with respect to time, we have

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = \frac{d}{dt}\left[m\dot{x}_1\right] = m\ddot{x}_1$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = \frac{d}{dt}\left[m\dot{x}_2\right] = m\ddot{x}_2 .$$

To determine the right side of each equation of motion, we differentiate the potential energy with respect to x_1 and x_2.

$$-\frac{\partial V}{\partial x_1} = -\frac{1}{2}k\left[4x_1 - 2x_2 + 0\right] = k\left[-2x_1 + x_2\right]$$

$$-\frac{\partial V}{\partial x_2} = -\frac{1}{2}k\left[0 - 2x_1 + 4x_2\right] = k\left[x_1 - 2x_2\right] .$$

Substituting this information into the equations of motion, we obtain

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = -\frac{\partial V}{\partial x_1} ,$$

which becomes

$$m\ddot{x}_1 = k\left[-2x_1 + x_2\right] \quad \text{or} \quad \ddot{x}_1 = \frac{k}{m}\left[-2x_1 + x_2\right]$$

and

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = -\frac{\partial V}{\partial x_2}$$

which becomes

$$m\ddot{x}_2 = k\left[x_1 - 2x_2\right] \quad \text{or} \quad \ddot{x}_2 = \frac{k}{m}\left[x_1 - 2x_2\right].$$

We now have a system of second order differential equations which can be written as the following matrix equation, where A is a symmetric matrix.

$$\vec{\ddot{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{k}{m} A \vec{X}$$

Since A is a symmetric matrix, all of its eigenvalues are real and A is diagonalizable. We begin the determination of the eigenvalues of the matrix A by

$$\det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = (-2-\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1).$$

If we set $\det(A - \lambda I)$ equal to zero and solve for λ, we find the eigenvalues are $\lambda = -3$ and $\lambda = -1$. Since A is diagonalizable there exists an invertible (orthogonal) matrix P such that $P^{-1}AP = D$. The matrix D is the diagonal matrix whose entries along the main diagonal consist of the eigenvalues of A and the columns of P are corresponding eigenvectors associated with these eigenvalues. To find P we need to find an eigenvector associated with $\lambda = -3$ and one associated with $\lambda = -1$. For $\lambda = -3$ we have the following matrix equation.

$$\begin{pmatrix} -2+3 & 1 \\ 1 & -2+3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{so} \quad x_1 = -x_2$$

If we let $x_1 = 1$, then $x_2 = -1$, and it follows that an eigenvector associated with the eigenvalue $\lambda = -3$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. To find an eigenvector associated with $\lambda = -1$, we use the following matrix equation.

$$\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{so} \quad x_1 = x_2$$

If we let $x_1 = 1$, then $x_2 = 1$. Thus, an eigenvector associated with the eigenvalue $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore, these two eigenvectors are the columns of the invertible matrix $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Recall that our goal is to solve the second order differential equation $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$. If we multiply both sides of this equation by P^{-1} and use the identity $P P^{-1} = I_2$, the 2×2 identity matrix, we obtain

$$(2.1) \qquad P^{-1}\ddot{\vec{X}} = P^{-1}\frac{k}{m} A \vec{X} = \frac{k}{m} P^{-1} A\left(P P^{-1}\right)\vec{X} = \frac{k}{m}\left(P^{-1} A P\right)\left(P^{-1}\vec{X}\right).$$

We want to get Equation (2.1) into a simpler form to make it easier to solve. To do this we will let $\vec{U} = P^{-1}\vec{X}$. This matrix equation can be easily differentiated with respect to time, since P^{-1} is a constant matrix. The first derivative with respect to time is $\dot{\vec{U}} = P^{-1}\dot{\vec{X}}$. Since Equation (2.1) is a second order differential equation, taking a second derivative with respect to time yields $\ddot{\vec{U}} = P^{-1}\ddot{\vec{X}}$. We introduce the vector variable $\vec{U}$ into Equation (2.1) by substituting $\ddot{\vec{U}} = P^{-1}\ddot{\vec{X}}$ into the left side of this equation. Then, if we substitute $P^{-1}AP = D$ and $\vec{U} = P^{-1}\vec{X}$ into the right side, Equation (2.1) becomes $\ddot{\vec{U}} = \frac{k}{m} D\,\vec{U}$. This system of second order differential equations is easier to solve than $\ddot{\vec{X}} = \frac{k}{m} A\,\vec{X}$.

Exercise 2.1

Show why the system $\ddot{\vec{U}} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{k}{m} D\,\vec{U}$ is easier to solve than $\ddot{\vec{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{k}{m} A\,\vec{X}$.

Recall that we are trying to solve for the vector $\vec{X}$. Rearranging $\vec{U} = P^{-1}\vec{X}$, we get $\vec{X} = P\,\vec{U}$, which tells us that instead of finding $\vec{X}$ we need only find $P\,\vec{U}$. Since we already know the matrix P, we must find the vector $\vec{U}$. If we multiply both sides of the matrix equation $\ddot{\vec{U}} = \frac{k}{m} D\,\vec{U}$ by the matrix P, we get $P\vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$. Rewriting the left side of $P\vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$, using the notation where $P^{(i)}$ represents the i th column (i = 1 and 2) of the matrix P, we obtain

$$P\vec{\ddot{U}} = \left(P^{(1)}\ P^{(2)}\right)\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = P^{(1)}\ddot{u}_1 + P^{(2)}\ddot{u}_2$$

Rewriting the right side of $P\vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$, we obtain

$$\frac{k}{m} P D \vec{U} = \frac{k}{m}\left[P^{(1)}\lambda_1 u_1 + P^{(2)}\lambda_2 u_2\right].$$

Thus $P\vec{\ddot{U}} = \frac{k}{m} P D \vec{U}$ can be written as

$$P^{(1)}\ddot{u}_1 + P^{(2)}\ddot{u}_2 = \frac{k}{m}\left[P^{(1)}\lambda_1 u_1 + P^{(2)}\lambda_2 u_2\right].$$

We can simplify this equation by multiplying through by $\frac{k}{m}$, gathering terms and moving all terms to the left side.

$$\left[P^{(1)}\ddot{u}_1 - \frac{k}{m} P^{(1)}\lambda_1 u_1\right] + \left[P^{(2)}\ddot{u}_2 - \frac{k}{m} P^{(2)}\lambda_2 u_2\right] = \vec{0}$$

Factoring out $P^{(1)}$ and $P^{(2)}$, we have

$$\left[\ddot{u}_1 - \frac{k}{m}\lambda_1 u_1\right]P^{(1)} + \left[\ddot{u}_2 - \frac{k}{m}\lambda_2 u_2\right]P^{(2)} = \vec{0}.$$

Since the columns of P are eigenvectors of A which correspond to distinct eigenvalues, we know they are linearly independent (in fact they are orthogonal). The equation produces a finite linear combination of linearly independent vectors which equals zero, thus the coefficients of $P^{(1)}$ and $P^{(2)}$ must be zero. If we set each of the coefficients in the equation above equal to zero, we obtain

$$\ddot{u}_1 - \frac{k}{m}\lambda_1 u_1 = 0 \quad \text{and} \quad \ddot{u}_2 - \frac{k}{m}\lambda_2 u_2 = 0.$$

These are both second order linear differential equations which can be solved using basic techniques. (See Appendix A.) If we let $r_i = -\frac{k}{m}\lambda_i$, where i=1, 2, then these equations become

$$\ddot{u}_1 + r_1 u_1 = 0 \quad \text{and} \quad \ddot{u}_2 + r_2 u_2 = 0.$$

Using the following formulas, we can solve for the vector $\vec{U}$. (Note: To determine whether r_i is zero, negative or positive, substitute λ_i into $r_i = -\frac{k}{m}\lambda_i$.)

If $r_i = 0$, then $u_i = c_{i1}t + c_{i2}$

If $r_i < 0$, then $u_i = c_{i1}e^{r_i t} + c_{i2}e^{-r_i t}$

If $r_i > 0$, then $u_i = c_{i1}\cos\left(\sqrt{r_i}\, t\right) + c_{i2}\sin\left(\sqrt{r_i}\, t\right)$.

The solution to the original system of differential equations $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$ is found by substituting the values for both the matrix P and the vector $\vec{U}$ into the equation $\vec{X} = P\,\vec{U}$.

Exercise 2.2

Using the above technique, solve the following system of differential

equations $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$. Where $\ddot{\vec{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}$, $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ and $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

That is, find the two equations which describe x_1 and x_2.

Exercise 2.3

Given a horizontal spring-weight system similar to Figure 2.3 with n blocks, the following equations would express the kinetic and potential energies of the system.

Kinetic Energy $\quad T = \frac{1}{2} m \sum_{i=1}^{n} \dot{x}_i^2$

Potential Energy $\quad V = \frac{1}{2} k \Big[b_{11} x_1^2 + b_{22} x_2^2 + \ldots + b_{nn} x_n^2$

$$+ 2b_{12} x_1 x_2 + 2b_{13} x_1 x_3 + \ldots + 2b_{1n} x_1 x_n$$

$$+ 2b_{23} x_2 x_3 + 2b_{24} x_2 x_4 + \ldots + 2b_{2n} x_2 x_n$$

$$\vdots$$

$$+ 2b_{i,\,i+1} x_i x_{i+1} + \ldots + 2b_{i,n} x_i x_n$$

$$\vdots$$

$$+ 2b_{n-2,\,n-1} x_{n-2} x_{n-1} + 2b_{n-2,\,n} x_{n-2} x_n$$

$$+ 2b_{n-1,\,n} x_{n-1} x_n \Big]$$

Use the equations of motion to find the system of differential equations which model the spring-weight system with n blocks. To solve this system, generalize the procedure used to solve the system of differential equations which model the spring-weight system with two blocks. (Hint: Some of the material that has been discussed can be used directly, while other portions will need some modifications.)

Another aspect of the spring-weight system that we want to consider is the oscillations of the system as a whole. From our work above, we know the solution to the second order differential equation $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$ can be found by using $\vec{X} = P\vec{U}$. In Exercise 2.2 we found

$$x_1 = u_1 + u_2 \text{ and } x_2 = -u_1 + u_2.$$

This is a system of two linear equations which we can solve for u_1 and u_2.

Thus we have the equations

$$(2.2) \qquad u_1 = \frac{x_1 - x_2}{2} \quad \text{and} \quad u_2 = \frac{x_1 + x_2}{2}$$

each of which gives a relationship between the variables x_1 and x_2. It is important to note that we could have found these equations directly from the matrix equation $\vec{U} = P^{-1}\vec{X}$, but this would involve finding P^{-1}. Using Figure 2.5, we can recall the configuration of this spring-weight system. Since the springs were stretched by differing amounts, a different frequency (the number of vibrations per unit time) is associated with each of the variables x_1 and x_2.

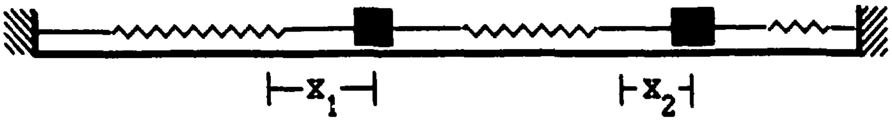


Figure 2.5

This spring-weight system has two separate modes in which it vibrates. In the first mode $u_1 = \frac{x_1 - x_2}{2}$ and $u_2 = 0$. Since $\frac{x_1 - x_2}{2}$ represents the how the distance between the two blocks is changing, the first mode of vibration describes how the distance between the two blocks is changing. For instance if x_2 is greater than x_1, then the change in distance between the blocks is smaller than the distance between the blocks when the spring-weight system is in equilibrium. However, if x_1 is greater than x_2, then the change in distance between the blocks is larger than the distance between the blocks when the spring-weight system is in equilibrium. Thus, the oscillation of the system in this mode is described by how the distance between the two blocks is changing which corresponds to the frequency associated with the second eigenvalue λ_2. To visualize this, consider the series of "snapshots" of the spring-weight system in motion in Figure 2.6, where the banner is made of an elastic material and indicates the distance between the two blocks in the system. When this system vibrates, we would see the banner contracting and stretching with a frequency associated with λ_2.

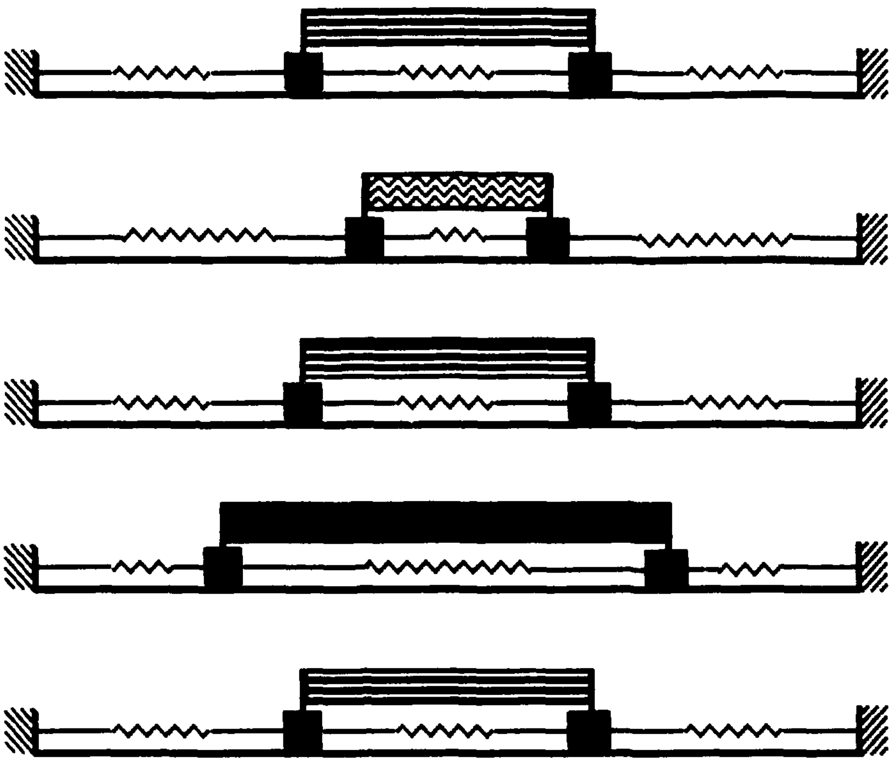


Figure 2.6

In the second mode $u_1=0$ and $u_2=\frac{x_1+x_2}{2}$. Since $\frac{x_1+x_2}{2}$ represents how the center of gravity of the system has changed, the second mode of vibrations describes the displacement of the center of gravity. Thus the oscillation of the system in this mode is where the center of gravity of the system vibrates at the frequency associated with the first eigenvalue λ_1.

To visualize this, consider the series of diagrams in Figure 2.7, where the flag indicates the center of gravity of the system. When this system vibrates, we see the flag moving back and forth with a frequency associated with λ_1. This is indicated by the following series of "snapshots" of the spring-weight system in motion.

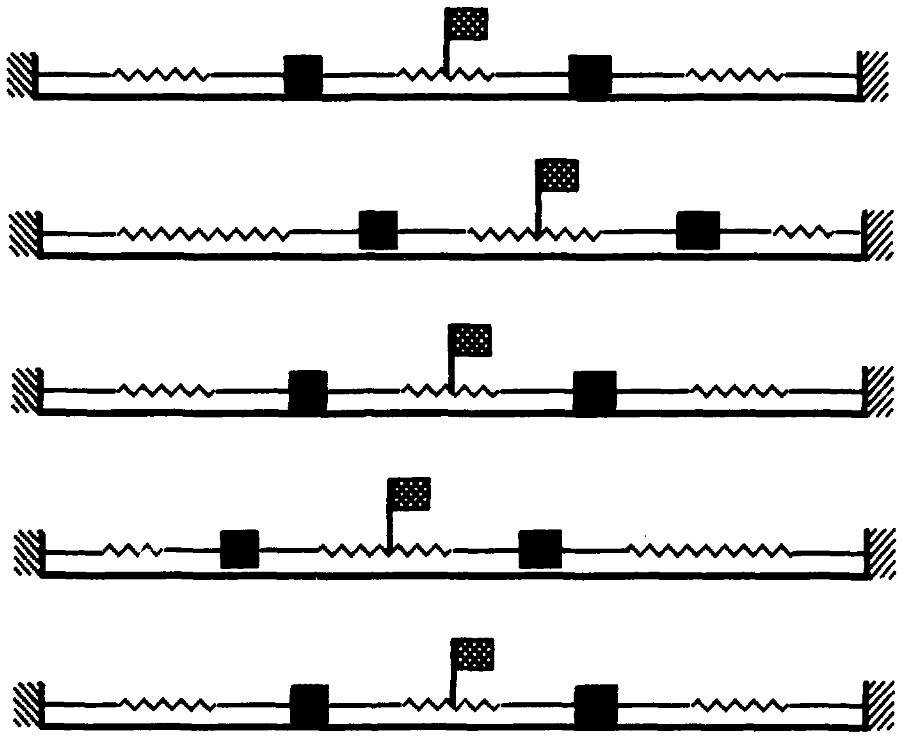


Figure 2.7

Exercise 2.4

Suppose the spring-weight system we have been studying was lying free in the xy-plane, that is, the ends of the springs are not anchored. Figure 2.8 can help us visualize this.

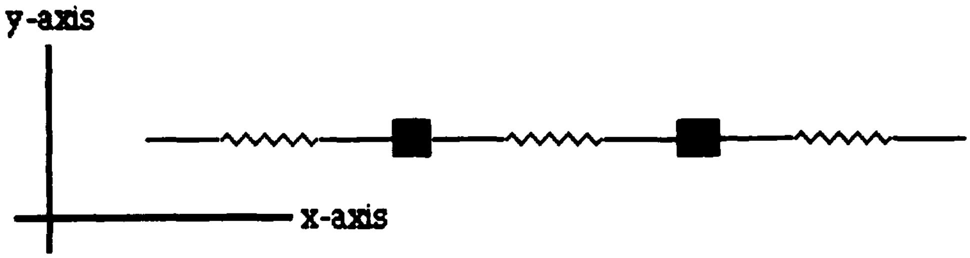


Figure 2.8

Using the information we have gained by studying the stationary spring-weight system, describe the motion (including the vibrations) that can occur. Note, there is no need to find the frequencies to complete this exercise. (Hint: consider other types of motion, besides vibrations.)

Exercise 2.5

(a) Determine the system of differential equations that model the motions of the spring-weight system given in Figure 2.9.

(b) Solve the system of differential equations.

(c) Describe the possible configurations in which it vibrates.

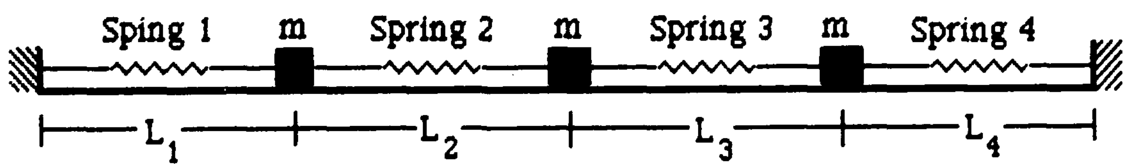


Figure 2.9

Exercise 2.6

Suppose the spring-weight system of Exercise 2.5 was lying free in the xy-plane, that is, the ends of the springs are not anchored. Describe the motion (including vibrations) that can occur for this system. Compare these motions with the motions found in Exercise 2.4.

Section 3 A Closed Spring-Weight System

In this section we will discuss how to mathematically model the spring-weight system in Figure 3.1 and determine the possible motions of the system. This system lies in the xy-plane with none of its blocks anchored. The mass of each of the three blocks is the same and is denoted by m. L is the length of each spring when the system is in its equilibrium configuration and k is the spring constant, which is the same for each spring.

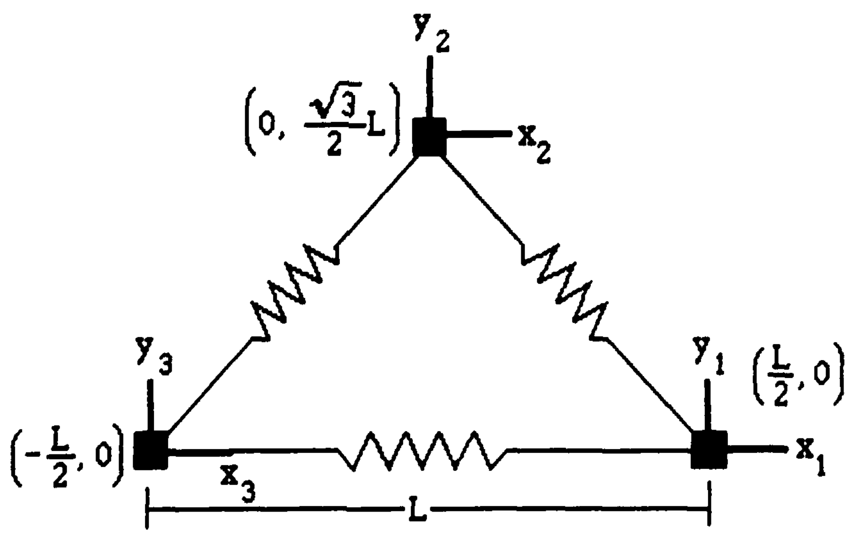


Figure 3.1

This system is in stable equilibrium when x_1-y_1-x_2-y_2-x_3-y_3-0. To find the energy of the system in Figure 3.1 we need to find the kinetic and potential energies of the system. Recall that the kinetic energy of the system is one half the mass times the sum of the square of the first derivative of each of the six variables with respect to time. Thus, the kinetic energy is

$$T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2 \right).$$

Finding the potential energy requires more work. Since the potential energy of the system is the sum of the potential energies of the springs, we first need to find the potential energy of each spring. We will consider each side of the triangle individually.

The first side that we look at is given in Figure 3.2.

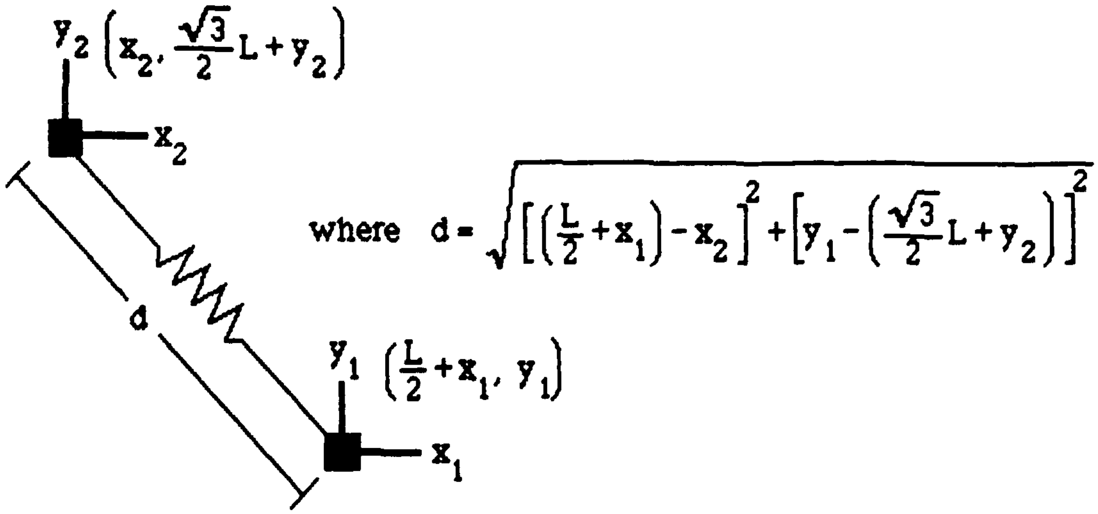


Figure 3.2

The potential energy of this spring is one half the spring constant k times the square of the distance that the spring is stretched. If we let d represent the length of the spring after it has been stretched, then the displacement of the spring from its equilibrium position (the distance that the spring is stretched) is $|d-L|$. Expressing the potential energy for the spring in terms of $|d-L|$, we have

$$V_{12} = \frac{1}{2}k|d - L|^2,$$

where the subscript 12 of V indicates that we are finding the potential energy of the spring that is stretched between the block with coordinates x_1 and y_1 to the block with coordinates x_2 and y_2. Now, we want to rewrite V_{12} using the variables x_1, y_1, x_2, and y_2. To do this, we must first simplify the expression for the distance d.

$$d = \left\{\left[\left(\frac{L}{2} + x_1\right) - x_2\right]^2 + \left[y_1 - \left(\frac{\sqrt{3}}{2}L + y_2\right)\right]^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left[\frac{L}{2} + \left(x_1 - x_2\right)\right]^2 + \left[-\frac{\sqrt{3}}{2}L + \left(y_1 - y_2\right)\right]^2\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1}{4}L^2 + L\left(x_1 - x_2\right) + \left(x_1 - x_2\right)^2 + \frac{3}{4}L^2 - \sqrt{3}\,L\left(y_1 - y_2\right) + \left(y_1 - y_2\right)^2\right\}^{\frac{1}{2}}$$

$$= L\left\{1 + \frac{1}{L}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right] + \frac{1}{L^2}\left[\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2\right]\right\}^{\frac{1}{2}}$$

We want to rewrite the quantity

$$\left\{1 + \frac{1}{L}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right] + \frac{1}{L^2}\left[\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2\right]\right\}^{\frac{1}{2}}$$

using its Taylor series expansion. The terms of higher powers have been grouped together for convenience.

$$d = L\left\{1 + \frac{\frac{1}{2}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right]}{1!}\frac{1}{L} + \text{terms of higher powers}\right\}$$

Exercise 3.1

Verify that the expression above is indeed the Taylor series expansion for the quantity $f\left(\frac{1}{L}\right)$ (shown below). Hint: write $f\left(\frac{1}{L}\right)$ in its Taylor series expanded about zero. Recall that L is the length of the spring in equilibrium, thus L≠0. (Hint: To make it easier to take the derivative of $f\left(\frac{1}{L}\right)$, let $r = \frac{1}{L}$ and find the derivative of f(r).)

$$f\left(\frac{1}{L}\right) = \left\{1 + \frac{1}{L}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right] + \frac{1}{L^2}\left[\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2\right]\right\}^{\frac{1}{2}}$$

The expression preceding Exercise 3.1 can be simplified by multiplying through by L and then moving L to the left side. The resulting quantity is what we want.

$$d - L = \frac{1}{2}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right] + \text{terms of higher powers}$$

This quantity can now be substituted into the formula for the potential energy V_{12}.

$$V_{12} = \frac{1}{2}k|d - L|^2$$

$$= \frac{k}{2}\left\{\frac{1}{4}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right) + \text{terms of higher powers}\right]^2\right\}$$

$$= \frac{k}{2}\left\{\frac{1}{4}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right]^2 + \text{terms of higher powers}\right\}$$

Recall from an earlier discussion that the Taylor series expansion for potential energy can not have any nonzero linear terms because we are in a system which has an equilibrium configuration. Also, we are only considering small vibrations so we ignore the terms of higher powers. The formula for potential energy V_{12} is

$$V_{12} = \frac{k}{2}\left\{\frac{1}{4}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right]^2\right\}$$

$$= \frac{k}{2}\left\{\frac{x_1^2}{4} + \frac{x_2^2}{4} + \frac{3y_1^2}{4} + \frac{3y_2^2}{4} - \frac{1}{2}x_1x_2 - \frac{\sqrt{3}}{2}x_1y_1 + \frac{\sqrt{3}}{2}x_2y_1 + \frac{\sqrt{3}}{2}x_1y_2 - \frac{\sqrt{3}}{2}x_2y_2 - \frac{3}{2}y_1y_2\right\}.$$

Exercise 3.2

Using Figure 3.3, find V_{23}. Hint: the procedure is similar to the one used to find V_{12}.

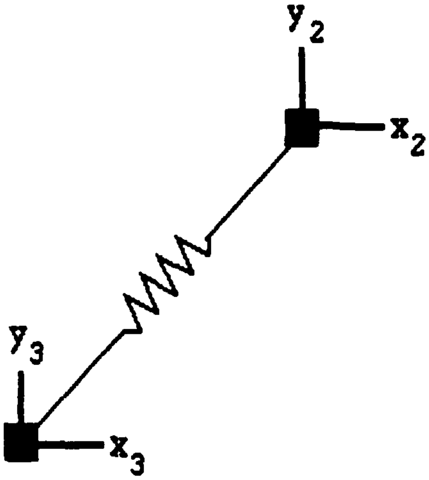


Figure 3.3

Exercise 3.3

Using Figure 3.4, find V_{13}.

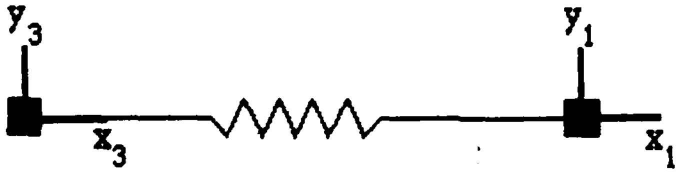


Figure 3.4

As stated before, the potential energy of the system is the sum of the potential energy of each spring. Thus we have

$$V = V_{12} + V_{23} + V_{13}$$

$$= \frac{k}{2}\Big\{ \frac{5}{4}x_1^2 + \frac{1}{2}x_2^2 + \frac{5}{4}x_3^2 + \frac{3}{4}y_1^2 + \frac{3}{2}y_2^2 + \frac{3}{4}y_3^2 - \frac{x_1x_2}{2} - \frac{x_2x_3}{2}$$

$$- 2x_1x_3 - \frac{3}{2}y_1y_2 - \frac{3}{2}y_2y_3 - \frac{\sqrt{3}}{2}x_1y_1 + \frac{\sqrt{3}}{2}x_2y_1 + \frac{\sqrt{3}}{2}x_1y_2$$

$$- \frac{\sqrt{3}}{2}x_3y_2 - \frac{\sqrt{3}}{2}x_2y_3 + \frac{\sqrt{3}}{2}x_3y_3\Big\}.$$

We recall that the kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right).$$

Since this system requires six coordinates to fully describe it, we know we must have six equations of motion. These are

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = -\frac{\partial V}{\partial x_1}, \qquad \left[\frac{\partial T}{\partial \dot{x}_2}\right] = -\frac{\partial V}{\partial x_2}, \qquad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_3}\right] = -\frac{\partial V}{\partial x_3},$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_4}\right] = -\frac{\partial V}{\partial x_4}, \qquad \left[\frac{\partial T}{\partial \dot{x}_5}\right] = -\frac{\partial V}{\partial x_5}, \quad \text{and} \quad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_6}\right] = -\frac{\partial V}{\partial x_6}.$$

First, we find the left side of each equation of motion, then find the right side and equate the two. Thus, we have the following six equations.

$$\ddot{x}_1 = \frac{k}{m}\left\{-\frac{5}{4}x_1 + \frac{1}{4}x_2 + x_3 + \frac{\sqrt{3}}{4}y_1 - \frac{\sqrt{3}}{4}y_2\right\}$$

$$\ddot{x}_2 = \frac{k}{m}\left\{\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{1}{4}x_3 - \frac{\sqrt{3}}{4}y_1 + \frac{\sqrt{3}}{4}y_3\right\}$$

$$\ddot{x}_3 = \frac{k}{m}\left\{x_1 + \frac{1}{4}x_2 - \frac{5}{4}x_3 + \frac{\sqrt{3}}{4}y_2 - \frac{\sqrt{3}}{4}y_3\right\}$$

$$\ddot{y}_1 = \frac{k}{m}\left\{\frac{\sqrt{3}}{4}x_1 - \frac{\sqrt{3}}{4}x_2 - \frac{3}{4}y_1 + \frac{3}{4}y_2\right\}$$

$$\ddot{y}_2 = \frac{k}{m}\left\{-\frac{\sqrt{3}}{4}x_1 + \frac{\sqrt{3}}{4}x_3 + \frac{3}{4}y_1 - \frac{3}{2}y_2 + \frac{3}{4}y_3\right\}$$

$$\ddot{y}_3 = \frac{k}{m}\left\{\frac{\sqrt{3}}{4}x_2 - \frac{\sqrt{3}}{4}x_3 + \frac{3}{4}y_2 - \frac{3}{4}y_3\right\}$$

This system of six equations can be written as a matrix equation. In order to eliminate fractions from the matrix, we factor $\frac{k}{4m}$ out of each equation, which results in $\frac{k}{4m}$ being factored out of the coefficient matrix A.

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} = \frac{k}{4m}\begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{k}{4m}A\,\vec{X}$$

The generalized theory developed in Exercise 2.3 describes the situation when n=6. Thus, we begin by finding the eigenvalues and eigenvectors of the symmetric matrix A. One way we could proceed would be to use the sixth degree characteristic polynomial to find the eigenvalues directly. However, this would require finding the determinant of a 6×6 matrix. Using the cofactor expansion method would require 6! or 720 calculations to find the value of the determinant. We could also use a computer program. For example, the user's guide to the computer program LINPACK (Dongarra, Bunch and Stewart, 1979) describes how the program can be used to approximate the eigenvalues and eigenvectors of the characteristic polynomial. This would be quicker, but would not give us any insight into the possible types of vibrations of the system. Instead, let us consider the symmetric matrix A and see if we can use our knowledge of matrices to reduce the amount of work required to find the eigenvalues. In general, the coefficient matrix which represents an application is much larger than a 6×6 matrix, but is still a symmetric matrix. The approach used by applied mathematicians working on large systems would be to: 1) manually work through the theory of a smaller, related, and less complicated system, 2) enlarge the system and use a computer to find the eigenvalues and eigenvectors, 3) interpret the physical meaning of the information from the computer by comparing the results with the results found in step 1, and finally, 4) change the model so that it reflects the desired system as closely as possible. For example, in a more complicated system not all of the blocks may be of the same mass, nor the springs be of the same length or have the same spring constant. Step 1 may be to consider a system where all of the blocks have the same mass, the springs are all of the same length, and each spring has the same spring constant.

Therefore, we will start our work by finding the determinant of the matrix A-λI

$$\det(A-\lambda I)=\begin{vmatrix} -5-\lambda & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2-\lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5-\lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3-\lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6-\lambda & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3-\lambda \end{vmatrix}.$$

Next, we replace the first row by the sum the first three rows and replace the last row by the sum of the last three rows to obtain the following interesting matrix.

$$\begin{vmatrix} -\lambda & -\lambda & -\lambda & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5-\lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3-\lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6-\lambda & 3 \\ 0 & 0 & 0 & -\lambda & -\lambda & -\lambda \end{vmatrix}=|A'|$$

If λ were set equal to zero, the matrix A', defined above, would have two rows of zeros indicating that λ=0 is an eigenvalue of A with multiplicity $\geq$ 2.

We pause for a moment in our pursuit of eigenvalues to find the eigenvectors associated with λ=0. To begin, λ=0 is substituted into A' so that the matrix equation $A'\vec{X} = \vec{0}$, which has been written in augmented form, can be solved.

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} & 0 \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Using Gaussian elimination, we reduce this system to a form that can easily be solved.

$$(3.1) \qquad \left(\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From this augmented system of equations, with three rows of zeros, we know three of the variables (x_3, y_2 and y_3) can take on any value, forcing the remaining three variable (x_1, x_2 and y_1) to take on specific values given by the following equations, obtained from the augmented matrix in Equation (3.1)..

$$x_1 = x_3$$

$$x_2 = x_3 - \sqrt{3}y_2 + \sqrt{3}y_3 \tag{3.2}$$

$$y_1 = 2y_2 - y_3$$

Thus, by letting x_3, y_2 and y_3 take on specific values, we will have three linearly independent eigenvectors. This means the eigenvalue $\lambda = 0$ must have multiplicity three.

Before we actually determine the values of the eigenvectors, let us pause for a moment to see how we can rewrite the potential energy function in a slightly different format which will help us to determine its value under certain conditions. Recall, the potential energy of the system is the sum of the potential energy of each spring. Thus we have

$$V = V_{12} + V_{23} + V_{13}$$

$$= \frac{k}{2}\Big\{ \frac{5}{4}x_1^2 + \frac{1}{2}x_2^2 + \frac{5}{4}x_3^2 + \frac{3}{4}y_1^2 + \frac{3}{2}y_2^2 + \frac{3}{4}y_3^2 - \frac{x_1x_2}{2} - \frac{x_2x_3}{2}$$

$$- 2x_1x_3 - \frac{3}{2}y_1y_2 - \frac{3}{2}y_2y_3 - \frac{\sqrt{3}}{2}x_1y_1 + \frac{\sqrt{3}}{2}x_2y_1 + \frac{\sqrt{3}}{2}x_1y_2$$

$$- \frac{\sqrt{3}}{2}x_3y_2 - \frac{\sqrt{3}}{2}x_2y_3 + \frac{\sqrt{3}}{2}x_3y_3\Big\}.$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \end{pmatrix} \frac{-k}{8m} \begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

We are able to rewrite the potential energy in this format because A is a symmetric matrix. We define a function which can be rewritten in this fashion as a **quadratic form**. Thus, Equation (3.3) is the potential energy expressed as a quadratic form.

$$V(\vec{X}) = \vec{X}^{T} \frac{-k}{8m} A \vec{X} \tag{3.3}$$

By the definition of an eigenvector $\vec{X}$, which is associated with the eigenvalue λ of the matrix A, we know that $A\vec{X} = \lambda\vec{X}$. If we let the three eigenvectors, associated with the eigenvalue $\lambda=0$, be represented by

$\vec{X}_{\lambda_1=0}$, $\vec{X}_{\lambda_2=0}$, and $\vec{X}_{\lambda_3=0}$, then

$$V\left(\vec{X}_{\lambda_j=0}\right)=\vec{X}^{T}_{\lambda_j=0}\frac{-k}{8m}A\,\vec{X}_{\lambda_j=0}=\vec{X}^{T}_{\lambda_j=0}\frac{-k}{8m}\lambda\,\vec{X}_{\lambda_j=0}=\vec{X}^{T}_{\lambda_j=0}\frac{-k}{8m}0\,\vec{X}_{\lambda_j=0}=0$$

for $j=1, 2$ or 3. This tells us that the potential energy is zero. We now examine the physical interpretation of zero potential energy.

To have zero potential energy in the system, all the springs must remain the same length L as in equilibrium. Thus, the only type of motion possible occurs when the entire system moves as a unit. This is called a **rigid motion**. Since the spring-weight system lies in the xy-plane, there are only two types of rigid motion: translations (movement in the x- or y-direction only) and rotations (the system pivots around its center of mass). These two motions can also be combined.

If we consider the vector $\vec{X}$, as a translation in the x-direction only, then the variables x_1, x_2 and x_3 must all change by the same value and the variables y_1, y_2 and y_3 can not change. We can express this vector as

$\begin{pmatrix} c \\ c \\ c \\ 0 \\ 0 \\ 0 \end{pmatrix}$. If we let c=1, an eigenvector associated with the eigenvalue λ=0 is

$\vec{X}_{\lambda_1=0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The graphical interpretation of $\vec{X}_{\lambda_1=0}$ can be seen in Figure 3.5.

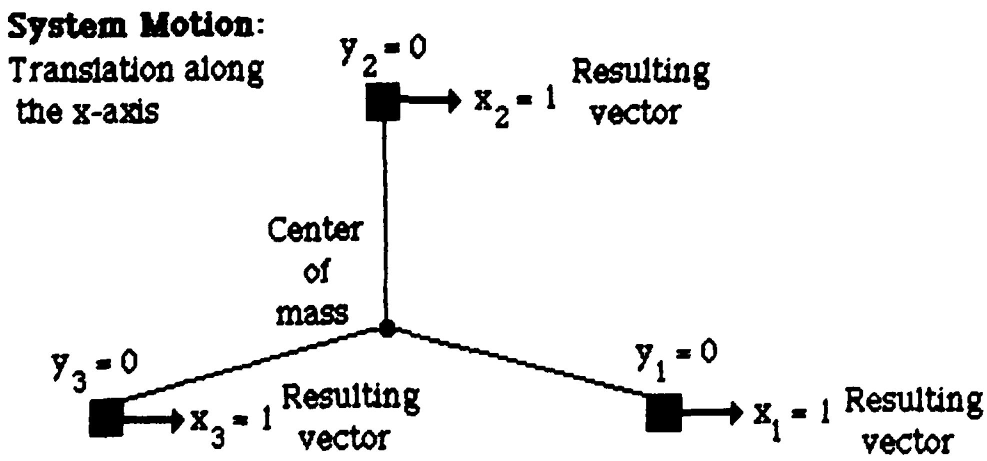


Figure 3.5

If the translation is to the right (in the positive x-direction), then c>0 and if it is to the left (in the negative x-direction), then c<0.

Exercise 3.4

Verify that $\vec{X}_{\lambda_1=0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a solution to

$$V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}^{T}_{\lambda_j=0} \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$$

The other type of translational motion we wish to consider occurs when the system moves in the y-direction only. The variables x_1, x_2 and x_3 do not change while the variables y_1, y_2 and y_3 must all change by the same value. We can express this vector as $\begin{pmatrix} 0 \\ 0 \\ 0 \\ c \\ c \\ c \end{pmatrix}$. If we let c=1, an eigenvector associated with the eigenvalue $\lambda=0$ is $\vec{X}_{\lambda_2=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. The

graphical interpretation of $\vec{X}_{\lambda_2=0}$ can be seen in

Figure 3.6.

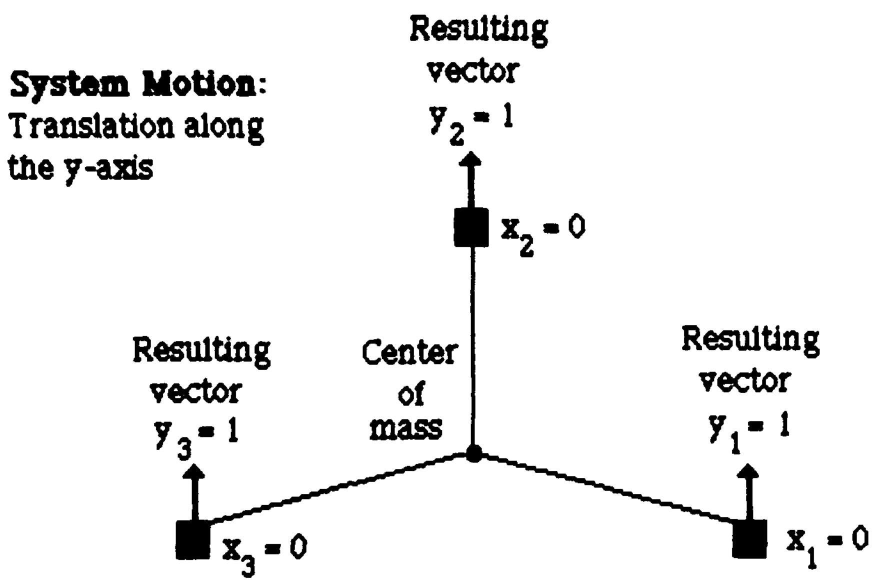


Figure 3.6

If the translation is upward (in the positive y-direction), then $c>0$ and if the translation is downward (in the negative y-direction), then $c<0$.

Exercise 3.5

Verify that $\vec{X}_{\lambda_2=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is a solution to $V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}^T_{\lambda_j=0} \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0.$

Taking a careful look at the vectors $\vec{X}_{\lambda_1=0}$ and $\vec{X}_{\lambda_2=0}$, we see they are orthogonal. That is, their dot product is zero. Since the eigenspace associated with the eigenvalue $\lambda=0$ has dimension at least three, we know there is a third linearly independent eigenvector associated with the eigenvalue $\lambda=0$. There are two ways we could proceed at this point. The first is to use the three equations in (3.2) and choose values for x_3, y_2 and y_3. For example, let $x_3=0$, $y_2=1$ and $y_3=0$, then use the Gram-Schmidt process to find a vector which is orthogonal to both $\vec{X}_{\lambda_1=0}$ and $\vec{X}_{\lambda_2=0}$. The other way is to replace two of the rows of zeros in the coefficient matrix in Equation (3.1) by the eigenvectors $\vec{X}^T_{\lambda_1=0}$ and $\vec{X}^T_{\lambda_2=0}$.

The solution to this new augmented matrix must satisfy all the equations which form the augmented matrix. Hence, the solution to the augmented system will satisfy both $x_1+x_2+x_3=0$ (from $\vec{X}_{\lambda_1=0}$) and $y_1+y_2+y_3=0$ (from $\vec{X}_{\lambda_2=0}$). A vector whose entries satisfy both of these equations is orthogonal to the eigenvectors $\vec{X}_{\lambda_1=0}$ and $\vec{X}_{\lambda_2=0}$. Also, from these two equations, we see in the solution to the equations associated with the augmented matrix, the x_i values must sum to zero. Therefore, there is no translational motion in the x-direction. Similarly, there is no translational motion in the y-direction. Thus, the center of mass does not move. Since this motion is a rigid motion ($\lambda=0$) and the center of mass of the system does not move, the rigid motion must be a rotation. Reducing the following augmented matrix which is Equation(3.1) with two of its rows of zeros replaced by $\vec{X}^T_{\lambda_1=0}$ and $\vec{X}^T_{\lambda_2=0}$

$$\left(\begin{array}{cccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \sqrt{3} & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

we obtain

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{2\sqrt{3}}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This augmented matrix above, can be interpreted as the following equations.

$$x_1 = -\frac{\sqrt{3}}{3} y_3$$

$$x_2 = \frac{2\sqrt{3}}{3} y_3$$

$$x_3 = -\frac{\sqrt{3}}{3} y_3$$

$$y_1 = -y_3$$

$$y_2 = 0$$

If we let $y_3=3$, then the resulting eigenvector is $\vec{X}_{\lambda_3=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$. The graphical interpretation of $\vec{X}_{\lambda_3=0}$ can be seen in Figure 3.7.

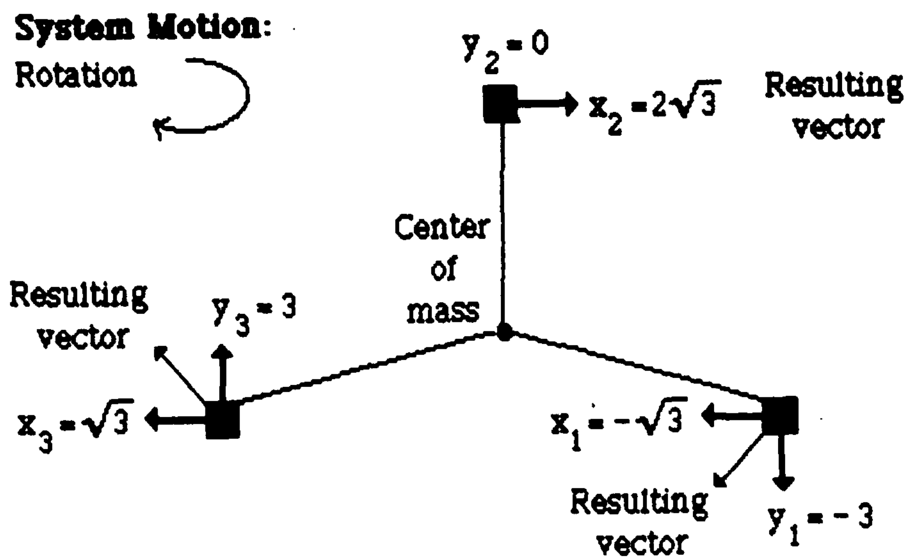


Figure 3.7

Exercise 3.6

Verify that $\vec{X}_{\lambda_3=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$ is a solution to

$$V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}^{T}_{\lambda_j=0} \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$$

Also, verify that $\vec{X}_{\lambda_3=0}$ is orthogonal to both $\vec{X}_{\lambda_1=0}$ and $\vec{X}_{\lambda_2=0}$.

So far we have found only three eigenvalues and their associated eigenvectors. The remaining three eigenvalues can be found using the determinant of the matrix A-λI which can be reduced to $|A'|$. For convenience, $|A'|$ has been repeated below.

$$\begin{vmatrix} -\lambda & -\lambda & -\lambda & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5-\lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3-\lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6-\lambda & 3 \\ 0 & 0 & 0 & -\lambda & -\lambda & -\lambda \end{vmatrix} = |A'|$$

Using Gaussian elimination, we will reduce the matrix to a form which will make the determinant easier to find. We will use only row (or column) operations that do not change the value of the determinant. After several row operations, we obtain

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3-\lambda & 0 & -2\sqrt{3} & -\sqrt{3} & 0 \\ 0 & -3 & -9-\lambda & \sqrt{3} & 2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} & -\sqrt{3} & -3-\lambda & 3 & 0 \\ 0 & -3\sqrt{3} & 0 & -6-2\lambda & -3-\lambda & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}.$$

At this point, we could find the determinant using the cofactor expansion method. However, if we do one column operation we will greatly reduce the number of calculations needed. We add -2 times the fifth column to the fourth column producing a new fourth column.

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3-\lambda & 0 & 0 & -\sqrt{3} & 0 \\ 0 & -3 & -9-\lambda & -3\sqrt{3} & 2\sqrt{3} & 0 \\ 0 & -2\sqrt{3} & -\sqrt{3} & -9-\lambda & 3 & 0 \\ 0 & -3\sqrt{3} & 0 & 0 & -3-\lambda & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

We are now ready to use the cofactor expansion method to find the determinant of the matrix A'. Expanding by the first column we have

$$|A'| = 1(-1)^{1+1}\begin{vmatrix} -3-\lambda & 0 & 0 & -\sqrt{3} & 0 \\ -3 & -9-\lambda & -3\sqrt{3} & 2\sqrt{3} & 0 \\ -2\sqrt{3} & -\sqrt{3} & -9-\lambda & 3 & 0 \\ -3\sqrt{3} & 0 & 0 & -3-\lambda & 0 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix}.$$

Now, expand the resulting cofactor by the fifth column.and obtain

$$|A'| = 1 \cdot 1(-1)^{5+5} \begin{vmatrix} -3-\lambda & 0 & 0 & -\sqrt{3} \\ -3 & -9-\lambda & -3\sqrt{3} & 2\sqrt{3} \\ -2\sqrt{3} & -\sqrt{3} & -9-\lambda & 3 \\ -3\sqrt{3} & 0 & 0 & -3-\lambda \end{vmatrix}.$$

Next expand the resulting cofactor by the first row.and obtain

$$|A'| = 1 \cdot 1 \left[(-3-\lambda)(-1)^{1+1} \begin{vmatrix} -9-\lambda & -3\sqrt{3} & 2\sqrt{3} \\ -\sqrt{3} & -9-\lambda & 3 \\ 0 & 0 & -3-\lambda \end{vmatrix} \right.$$

$$\left. + -\sqrt{3}(-1)^{1+4} \begin{vmatrix} -3 & -9-\lambda & -3\sqrt{3} \\ -2\sqrt{3} & -\sqrt{3} & -9-\lambda \\ -3\sqrt{3} & 0 & 0 \end{vmatrix} \right]$$

$$= \left[(-9-\lambda)^2 - 9\right]\left[(-3-\lambda)^2 - 9\right].$$

To find the eigenvalues of the original matrix, we set each factor equal to zero and solve for λ.

$$(-9-\lambda)^2 - 9 = 0 \qquad (-3-\lambda)^2 - 9 = 0$$

$$(-9-\lambda)^2 = 9 \qquad (-3-\lambda)^2 = 9$$

$$-9-\lambda = \pm 3 \qquad -3-\lambda = \pm 3$$

$$\lambda = -12, \ -6 \qquad \lambda = -6, \ 0$$

Since we have already determined that the eigenvalue $\lambda=0$ has multiplicity of at least three, the fact $\lambda=0$ occurs above should be no surprise. The remaining eigenvalues for the matrix A are $\lambda=-12$ and $\lambda=-6$, the latter with multiplicity 2.

To help us find the associated eigenvectors for the remaining three eigenvalues, we recall the equation $A\vec{X}_\lambda = \lambda\vec{X}_\lambda$ from the definition of an eigenvector $\vec{X}_\lambda$, and Equation 3.3 $V\left(\vec{X}_\lambda\right) = \vec{X}_\lambda^{\,T}\frac{-k}{8m}A\vec{X}_\lambda$, which describes the potential energy as a quadratic form using eigenvectors. As we saw earlier, these two equations can be combined as $V\left(\vec{X}_\lambda\right) = \vec{X}_\lambda^{\,T}\frac{-k}{8m}\lambda\vec{X}_\lambda$. We observe that the only way this equation can equal zero is if $\lambda=0$ or $\vec{X}_\lambda$ is the zero vector. However, since we are only looking at $\lambda=-12$ or $\lambda=-6$, which are nonzero values, we must have that $\vec{X}_\lambda$ be the zero vector in order for $V\left(\vec{X}_\lambda\right)$ to equal zero. Clearly this cannot happen because $\vec{X}_\lambda$ is an eigenvector which by definition is never equal to the zero vector. This indicates that the potential energy of the system is not zero. Hence, the potential energy of each spring is not zero, so the length of at least one of the springs must change. Thus, we do not have a rigid motion. Also, we recall that the determinant of the matrix $A-\lambda I$ can be reduced by summing the first three rows and the last three rows to give $|A'|$.

$$\begin{vmatrix} -\lambda & -\lambda & -\lambda & 0 & 0 & 0 \\ 1 & -2-\lambda & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5-\lambda & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3-\lambda & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6-\lambda & 3 \\ 0 & 0 & 0 & -\lambda & -\lambda & -\lambda \end{vmatrix} = |A'|$$

If we substitute in $\lambda=-12$ or $\lambda=-6$, the first row will contain constant values for x_1, x_2, and x_3 and the last row will contain constant values for y_1, y_2, and y_3. From an earlier discussion (following Exercise 3.5) this indicates there is no translational motion in either the x- or y-directions, so we know the center of mass does not move. Thus the motion associated with the last three eigenvectors can be thought of as vibrations of the blocks (but not a translation or rotation) with the center of mass remaining fixed.

First, we find the two linearly independent eigenvectors associated with the eigenvalue $\lambda=-6$. Since $\lambda=-6$ has multiplicity two, the solution space of the augmented matrix, $(A-\lambda I)\vec{X}=\vec{0}$ or $(A+6I)\vec{X}=\vec{0}$ will have dimension four. That is, when the augmented system is reduced, we will have two rows of zeros. Thus, four of the variables can be written in terms of two of the other variables. These two variables can be assigned values which will produce two linearly independent eigenvectors. If we let x_2 and y_2 be these two variables, then x_1, y_1, x_3 and y_3 can be written in terms of x_2 and y_2. One way to assign values to x_2 and y_2 and be assured of getting a linearly independent eigenvector, is to first let $x_2=0$ and $y_2=1$, and then let $x_2=1$ and $y_2=0$. Let us consider the geometric interpretation of these cases.

CASE 1. $x_2=0$ and $y_2=1$

Since the center of mass for this configuration remains fixed, the y_2 component must be balanced by the sum of the y_1 and the y_3 components. Because $x_2=0$, we know the components x_1 and x_3 must be equal in magnitude and of opposite sign. These components can be seen in Figure 3.8.

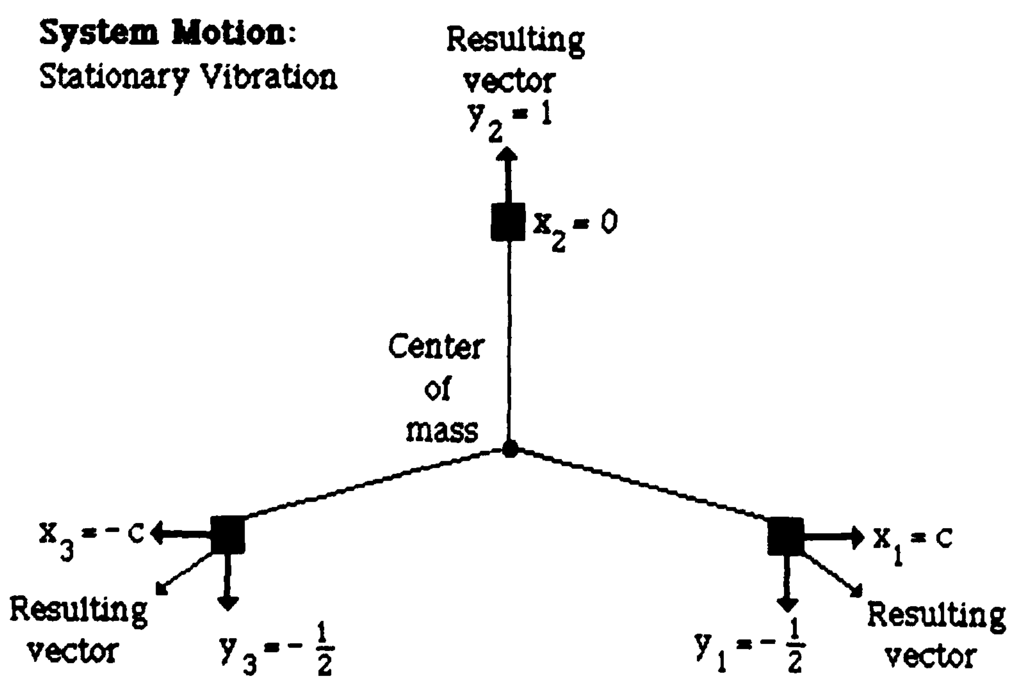


Figure 3.8

Letting c=1, one eigenvector associated with $\lambda=-6$ is $\vec{X}_{\lambda_1=-6} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$.

CASE 2. $x_2=1$ and $y_2=0$

Since the center of mass does not move, the x_2 component is balanced by the sum of the x_1 and x_3 components. Because $y_2=0$, we know the components y_1 and y_3 must be equal in magnitude and of opposite sign.

These components can be seen in Figure 3.9.

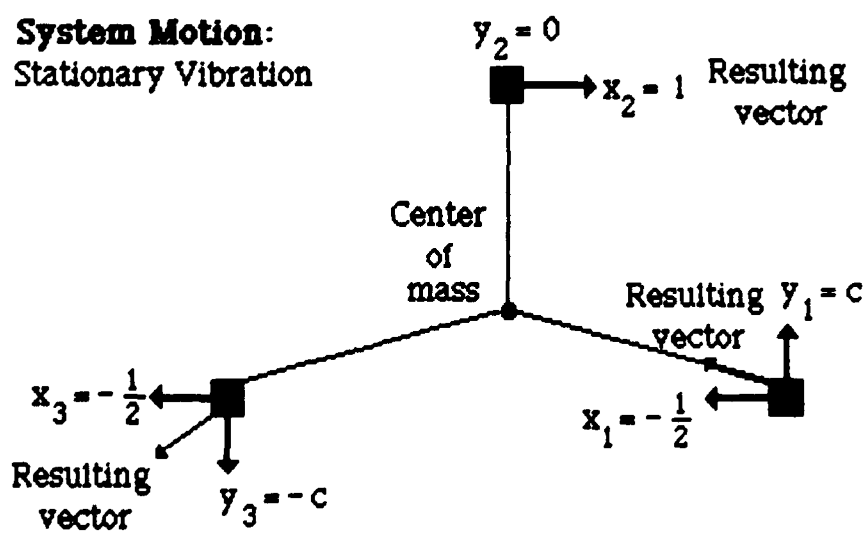


Figure 3.9

Letting c=1, a second eigenvector associated with $\lambda=-6$ is $\vec{X}_{\lambda_2=-6} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ -1 \end{pmatrix}$.

Exercise 3.7

Show that $\vec{X}_{\lambda_1=-6}$ and $\vec{X}_{\lambda_2=-6}$ are orthogonal (their dot product is zero).

Therefore, { $\vec{X}_{\lambda_1=-6}$, $\vec{X}_{\lambda_2=-6}$ } is a set of orthogonal eigenvectors associated with the eigenvalue $\lambda=-6$.

It remains for us to find the single eigenvector associated with the eigenvalue $\lambda=-12$. To do this we will substitute -12 for λ in $|A'|$ and solve the matrix equation $A'\vec{X} = \vec{0}$. When we do this, we get the following augmented matrix.

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This augmented matrix above, can be interpreted as the following equations.

$$x_1 = -\sqrt{3}y_3$$

$$x_2 = 0$$

$$x_3 = \sqrt{3}y_3$$

$$y_1 = y_3$$

$$y_2 = -2y_3$$

If we let $y_3 = 1$, then the resulting eigenvector is $\vec{X}_{\lambda=-12} = \begin{pmatrix} -\sqrt{3} \\ 0 \\ \sqrt{3} \\ 1 \\ -2 \\ 1 \end{pmatrix}$. The graphical interpretation of $\vec{X}_{\lambda=-12}$ can be seen in Figure 3.10.

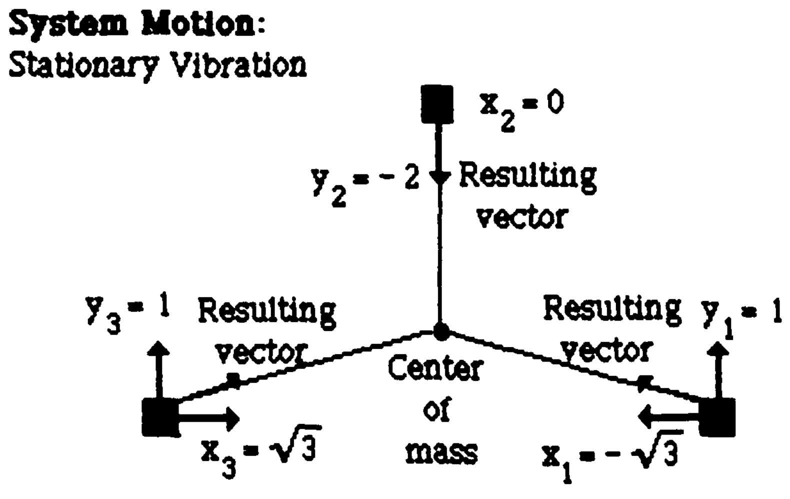


Figure 3.10

Thus, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1/2 \\ 1 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ 0 \\ \sqrt{3} \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ is an orthogonal

set of eigenvectors of A associated with the eigenvalues 0,0,0,-6,-6 and -12, respectively. We normalize these orthogonal vectors to get the following orthonormal set of eigenvectors.

$$\left\{ \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{14}}{2} \\ 0 \\ -\frac{\sqrt{14}}{2} \\ -\frac{\sqrt{14}}{4} \\ \frac{\sqrt{14}}{2} \\ -\frac{\sqrt{14}}{4} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{14}}{4} \\ \frac{\sqrt{14}}{2} \\ -\frac{\sqrt{14}}{4} \\ \frac{\sqrt{14}}{2} \\ 0 \\ -\frac{\sqrt{14}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{6} \end{pmatrix} \right\}$$

If we let the eigenvectors above form the columns of a matrix P, then P is invertible and $P^{-1}AP=D$, so $P^{-1}\frac{k}{4m}AP = \frac{k}{4m}D$ where D is a diagonal matrix with the eigenvalues 0, 0, 0, -6, -6, and -12 as the entries on the diagonal.

Since our goal is to solve the differential equation $\ddot{\vec{X}} = \frac{k}{4m} A \vec{X}$, we will let $\vec{U} = P^{-1}\vec{X}$, then apply Exercise 2.4 with n=6, where we have factored $\frac{k}{4m}$ out of the matrix A instead of $\frac{k}{m}$. Thus $P\ddot{\vec{U}} = \frac{k}{4m} P D \vec{U}$ becomes

$$P^{(1)}\ddot{u}_1 + P^{(2)}\ddot{u}_2 + \ldots + P^{(6)}\ddot{u}_6 = \frac{k}{4m}\left[P^{(1)}\lambda_1 u_1 + P^{(2)}\lambda_2 u_2 + \ldots + P^{(6)}\lambda_6 u_6\right],$$

where $\lambda_1=\lambda_2=\lambda_3=0$, $\lambda_4=\lambda_5=-6$, and $\lambda_1=-12$. This equation can be simplified by multiplying through by $\frac{k}{4m}$, gathering terms and moving everything to the left side.

$$\left[P^{(1)}\ddot{u}_1 - \frac{k}{4m}P^{(1)}\lambda_1 u_1\right] + \left[P^{(2)}\ddot{u}_2 - \frac{k}{4m}P^{(2)}\lambda_2 u_2\right] + \ldots$$

$$+\left[P^{(6)}\ddot{u}_6 - \frac{k}{4m}P^{(6)}\lambda_6 u_6\right] = \vec{0}$$

Factoring out $P^{(1)}, P^{(2)}, \ldots, P^{(6)}$, we have

$$\left[\ddot{u}_1 - \frac{k}{4m}\lambda_1 u_1\right]P^{(1)} + \left[\ddot{u}_2 - \frac{k}{4m}\lambda_2 u_2\right]P^{(2)} + \ldots + \left[\ddot{u}_6 - \frac{k}{4m}\lambda_6 u_6\right]P^{(6)} = \vec{0}.$$

Since the columns of P are orthonormal eigenvectors of A, we know they are linearly independent. Thus, we have a finite linear combination of linearly independent vectors which equals zero, so the coefficients of $P^{(1)}, P^{(2)}, \ldots, P^{(2)}$ must be zero. If we set each of the coefficients in the above equation equal to zero, we have

$$\begin{array}{ccc} \ddot{u}_1 - \frac{k}{4m}\lambda_1 u_1 = 0 & & \ddot{u}_1 = \frac{k}{4m}\lambda_1 u_1 \\ \ddot{u}_2 - \frac{k}{4m}\lambda_2 u_2 = 0 & \text{or} & \ddot{u}_2 = \frac{k}{4m}\lambda_2 u_2 \\ \vdots & & \vdots \\ \ddot{u}_6 - \frac{k}{4m}\lambda_6 u_6 = 0 & & \ddot{u}_6 = \frac{k}{4m}\lambda_6 u_6 \end{array}.$$

Since $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = \lambda_5 = -6$ and $\lambda_6 = -12$, these differential equations become

$$\ddot{u}_1 = 0 \qquad \ddot{u}_4 = -\frac{3k}{2m} u_4$$

$$\ddot{u}_2 = 0 \qquad \ddot{u}_5 = -\frac{3k}{2m} u_5$$

$$\ddot{u}_3 = 0 \qquad \ddot{u}_6 = -\frac{3k}{m} u_6 \quad .$$

These are all second order linear differential equations which can be solved using basic techniques. (See Appendix A.) The solutions are

$$u_1 = c_{11}t + c_{12}$$

$$u_2 = c_{21}t + c_{22}$$

$$u_3 = c_{31}t + c_{32}$$

$$u_4 = c_{41}\cos\left(\sqrt{\frac{3k}{2m}}\, t\right) + c_{42}\sin\left(\sqrt{\frac{3k}{2m}}\, t\right)$$

$$u_5 = c_{51}\cos\left(\sqrt{\frac{3k}{2m}}\, t\right) + c_{52}\sin\left(\sqrt{\frac{3k}{2m}}\, t\right)$$

$$u_6 = c_{61}\cos\left(\sqrt{\frac{3k}{m}}\, t\right) + c_{62}\sin\left(\sqrt{\frac{3k}{m}}\, t\right) \quad .$$

These solutions can be written as a matrix equation which can be substituted into $\vec{X} = P\,\vec{U}$.

$\vec{X} = P\,\vec{U}$

$$= \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{6} & \frac{\sqrt{14}}{2} & \frac{\sqrt{14}}{4} & -\frac{1}{2} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{14}}{2} & 0 \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{14}}{2} & -\frac{\sqrt{14}}{4} & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{1}{2} & -\frac{\sqrt{14}}{4} & \frac{\sqrt{14}}{2} & \frac{\sqrt{3}}{6} \\ 0 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{14}}{2} & 0 & -\frac{\sqrt{3}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{1}{2} & -\frac{\sqrt{14}}{4} & -\frac{\sqrt{14}}{2} & \frac{\sqrt{3}}{6} \end{pmatrix} \begin{pmatrix} c_{11}t + c_{12} \\ c_{21}t + c_{22} \\ c_{31}t + c_{32} \\ c_{41}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right) + c_{42}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) \\ c_{51}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right) + c_{52}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) \\ c_{61}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) + c_{62}\sin\left(\sqrt{\frac{3k}{m}}\,t\right) \end{pmatrix}.$$

The solution $\vec{X}$ to $\ddot{\vec{X}} = \frac{k}{4m} A\,\vec{X}$ is found by multiplying the matrix P by the vector $\vec{U}$. The components of the solution vector $\vec{X}$ are

$$x_1 = c_{11}\frac{\sqrt{3}}{3}t + c_{12}\frac{\sqrt{3}}{3} - c_{31}\frac{\sqrt{3}}{6}t - c_{32}\frac{\sqrt{3}}{6} + c_{41}\frac{\sqrt{14}}{2}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$+ c_{42}\frac{\sqrt{14}}{2}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) - c_{51}\frac{\sqrt{14}}{4}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{52}\frac{\sqrt{14}}{4}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) - c_{61}\frac{1}{2}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) - c_{62}\frac{1}{2}\sin\left(\sqrt{\frac{3k}{m}}\,t\right)$$

$$x_2 = c_{11}\frac{\sqrt{3}}{3}t + c_{12}\frac{\sqrt{3}}{3} + c_{31}\frac{\sqrt{3}}{3}t + c_{32}\frac{\sqrt{3}}{3} + c_{51}\frac{\sqrt{14}}{2}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{52}\frac{\sqrt{14}}{2}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$x_3 = c_{11}\frac{\sqrt{3}}{3}t + c_{12}\frac{\sqrt{3}}{3} - c_{31}\frac{\sqrt{3}}{6}t - c_{32}\frac{\sqrt{3}}{6} + c_{41}\frac{\sqrt{14}}{2}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{42}\frac{\sqrt{14}}{2}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) - c_{51}\frac{\sqrt{14}}{4}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{52}\frac{\sqrt{14}}{4}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) + c_{61}\frac{1}{2}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) + c_{62}\frac{1}{2}\sin\left(\sqrt{\frac{3k}{m}}\,t\right)$$

$$y_1 = c_{21}\frac{\sqrt{3}}{3}t + c_{22}\frac{\sqrt{3}}{3} - c_{31}\frac{1}{2}t - c_{32}\frac{1}{2} + c_{41}\frac{\sqrt{14}}{4}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{42}\frac{\sqrt{14}}{4}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) + c_{51}\frac{\sqrt{14}}{2}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$+ c_{52}\frac{\sqrt{14}}{2}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) + c_{61}\frac{\sqrt{3}}{6}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) + c_{62}\frac{\sqrt{3}}{6}\sin\left(\sqrt{\frac{3k}{m}}\,t\right)$$

$$y_2 = c_{21}\frac{\sqrt{3}}{3}t + c_{22}\frac{\sqrt{3}}{3} + c_{41}\frac{\sqrt{14}}{2}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right) + c_{42}\frac{\sqrt{14}}{2}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{61}\frac{\sqrt{3}}{3}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) - c_{62}\frac{\sqrt{3}}{3}\sin\left(\sqrt{\frac{3k}{m}}\,t\right)$$

$$y_3 = c_{21}\frac{\sqrt{3}}{3}t + c_{22}\frac{\sqrt{3}}{3} + c_{31}\frac{1}{2}t + c_{32}\frac{1}{2} - c_{41}\frac{\sqrt{14}}{4}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{42}\frac{\sqrt{14}}{4}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) - c_{51}\frac{\sqrt{14}}{2}\cos\left(\sqrt{\frac{3k}{2m}}\,t\right)$$

$$- c_{52}\frac{\sqrt{14}}{2}\sin\left(\sqrt{\frac{3k}{2m}}\,t\right) + c_{61}\frac{\sqrt{3}}{6}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) + c_{62}\frac{\sqrt{3}}{6}\sin\left(\sqrt{\frac{3k}{m}}\,t\right).$$

It is possible to determine the values of the c_{ij} provided we have been given enough details about the system which we have modeled. Since the values of m (the mass of the spring) and L (the length of the spring in equilibrium) are given, and if we know the value of each c_{ij}, then we will be able to find the value for each x_i and y_i at a given time.

We now want to apply the theory of the spring-weight system which we have just studied to understand how this system models the vibrations of a water or H_2O molecule, as shown in Figure 3.11.

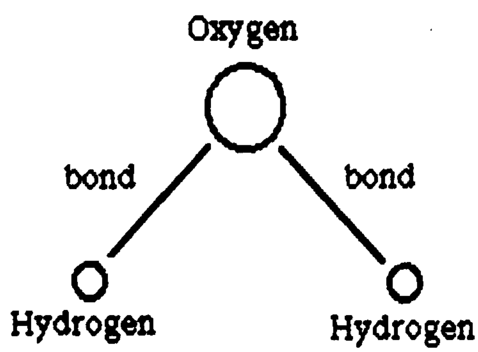


Figure 3.11

Figure 3.1 has three springs, but the water molecule has only two bonds. The third spring represents the repulsion force of the two hydrogen atoms. A water molecule which lies in the xy-plane would have a translational motion in both the x- and y-directions as described in Figures 3.5 and 3.6. Also, the molecule would be able to rotate, as we saw in Figure 3.7.

Moncrief and Jones (1977) explain the three vibrational modes for H_2O using Figure 3.12.

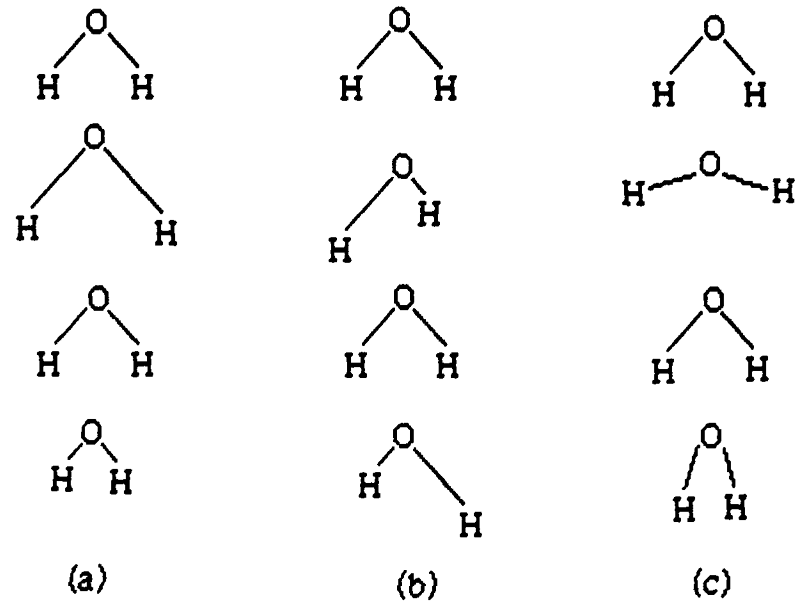


Figure 3.12

The vibration in Figure 3.12(a) is called a symmetric stretch since the bonds between the both hydrogen atoms and the oxygen atom are stretching by the same amounts at the same time. We have already consider this type of motion in Figure 3.8. In Figure 3.12(b) we see an asymmetric stretch which is due to the fact the bonds between the hydrogen atoms and the oxygen atom are being stretched by the same amount but not at the same time. Figure 3.9 describes this same mode of vibration in the closed spring-weight system. The final mode of vibration is called symmetric bending and is seen in Figure 3.12(c). This vibrational mode consists of the hydrogen-oxygen bonds remaining at the same length.

but the two hydrogen atoms vibrate by moving further apart then closer together. We have already seen this in Figure 3.10.

References

Dongarra, J., Bunch, J., & Stewart, C. (1979). LINPACK user's guide [Computer program manual]. Philadelphia, PA: SIAM.

Moncrief, J. W., & Jones, W. H. (1977). Elements of physical chemistry. Reading, MA: Addison-Wesley.

Application II Appendix A: Review of Differential Equations

We will limit our discussion to second order linear differential equations with constant coefficients. This appendix is not meant to replace a differential equation course, but only to show how to solve a very select group of differential equations. The second order linear differential equations which we want to solve are of the form

$$\ddot{x} + m\,x = 0, \tag{A.1}$$

where the coefficient of the x-term is a constant which we denote by m. Any second order differential equation which can be put in the form of Equation (A.1) is called a **linear differential equation**. The differential equation $\ddot{x} + m\,\sin x = 0$ is no longer linear because $\sin x$ is a nonlinear function of x. The method used to solve the differential equation (A.1) above, depends on the value of m. We will consider three possible cases.

CASE 1. m=0

If m=0, then our second order differential equation becomes

$$\ddot{x} = 0.$$

By the Fundamental Theorem of Calculus, if $\ddot{x} = 0$, then $\dot{x} = a$, and $x = at + b$.

Conversely, if $x = at + b$, then differentiating this equation with respect to time we have

$$\frac{dx}{dt} = a \quad \text{or} \quad \dot{x} = a$$

Differentiating again, we obtain

$$\frac{d^2x}{dt^2} = 0 \quad \text{or} \quad \ddot{x} = 0$$

Therefore, we conclude that $x = at + b$ is the solution to the differential equation $\ddot{x} = 0$.

CASE 2. m<0

The differential equation $\ddot{x} + m\,x = 0$ can be rearranged as $\ddot{x} = -m\,x$ where -m is a positive number. Recall from calculus, that the exponential function, when differentiated, yields a multiple of itself. Thus, we want an exponential function which when differentiated twice results in a positive multiple of itself. Let us pause for a moment and consider two examples of exponential functions.

$$x = e^{2t} \quad \text{and} \quad x = e^{-2t}$$

Taking the first derivative of these two function with respect to time, we obtain

$$\dot{x} = 2e^{2t} \quad \text{and} \quad \dot{x} = -2e^{-2t}.$$

After taking the second derivative, we have the following two functions

$$\ddot{x} = 4e^{2t} \quad \text{and} \quad \ddot{x} = 4e^{-2t}.$$

If we substitute the values for $\ddot{x}$ and x into the differential equation $\ddot{x} - 4x = 0$, we see that $x = e^{2t}$ and $x = e^{-2t}$ are both solutions to the same differential equation. Furthermore, any linear combination of these two solutions such as $x = c_1e^{2t} + c_2e^{-2t}$, is also a solution to $\ddot{x} + mx = 0$ when m--4. From this we conclude that $x = c_1e^{\sqrt{-m}\,t} + c_2e^{-\sqrt{-m}\,t}$, is a solution, for all c_1 and c_2.

CASE 3. m>0

The differential equation $\ddot{x} + mx = 0$ can be rearranged as $\ddot{x} = -mx$ where -m is a negative number. Recall from calculus, that the cosine function, when differentiated twice yields a negative multiple of itself. This is also true for the sine function. Let us pause for a moment and consider two examples involving the cosine and sine functions.

$$x = \cos 2x \quad \text{and} \quad x = \sin 2x$$

Taking the first derivative of these two function with respect to time, we obtain

$$\dot{x} = -2\sin 2x \quad \text{and} \quad \dot{x} = 2\cos 2x.$$

After taking the second derivative, we have the following two functions

$$\ddot{x} = -4\cos 2x \quad \text{and} \quad \ddot{x} = -4\sin 2x.$$

If we substitute the values for $\ddot{x}$ and x into the differential equation $\ddot{x} + 4x = 0$, we see that $x = \cos 2x$ and $x = \sin 2x$ are both solutions to the same differential equation. Furthermore, any linear combination of these two solutions such as $x = c_1 \cos 2x + c_2 \sin 2x$, is also a solution to $\ddot{x} + mx = 0$ when m=4. From this we conclude that $x = c_1 \cos \sqrt{m}\, t + c_2 \sin \sqrt{m}\, t$, is a solution, for all c_1 and c_2.

Just as it was shown in Case 1, where m=0, every solution of $\ddot{x} = 0$ must be in the form at+b. It can also be shown that every solution for Cases 2 and 3, where m=0, must be in the forms we have presented.

Exercise 2.1

The system $\ddot{\vec{U}} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{k}{m} D \vec{U}$ can be rewritten as the following two second order differential equations

(B.1)
$$\ddot{u}_1 = \frac{-3k}{m} u_1$$
$$\ddot{u}_2 = \frac{-k}{m} u_2 .$$

Similarly, the system $\ddot{\vec{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{k}{m} A \vec{X}$ can be rewritten as the following two second order differential equations

(B.2)
$$\ddot{x}_1 = \frac{-2k}{m} x_1 + \frac{k}{m} x_2$$
$$\ddot{x}_2 = \frac{k}{m} x_1 - \frac{2k}{m} x_2 .$$

Each equation in (B.1) can be solved independently using only basic techniques from differential equations. However, since each equation in (B.2) is in terms of both variables x_1 and x_2, neither equation can be solved independently. Thus, it is much easier to solve the system of differential equations given by $\ddot{\vec{U}} = \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{k}{m} D \vec{U}$.

Exercise 2.2

We have already found the eigenvalues of A to be $\lambda_1=-3$ and $\lambda_2=-1$, thus,

$r_1=\frac{3k}{m}$ and $r_2=\frac{1k}{m}$, which are both greater than zero. From this we see that the two second order differential equations in u_1 and u_2 are

$$\ddot{u}_1+\frac{3k}{m}u_1=0 \quad \text{and} \quad \ddot{u}_2+\frac{1k}{m}u_2=0 .$$

These are both second order linear differential equations which can be solved using basic techniques. Their solutions are

$$u_1=c_{11}\cos\left(\sqrt{\frac{3k}{m}}\,t\right)+c_{12}\sin\left(\sqrt{\frac{3k}{m}}\,t\right)$$

$$u_2=c_{21}\cos\left(\sqrt{\frac{1k}{m}}\,t\right)+c_{22}\sin\left(\sqrt{\frac{1k}{m}}\,t\right) .$$

The solution to the original system of differential equations $\ddot{\vec{X}}=\frac{k}{m}A\vec{X}$ is found by substituting the values for both the matrix P and the vector $\vec{U}$ into the equation $\vec{X}=P\,\vec{U}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} c_{11}\cos\left(\sqrt{\frac{3k}{m}}\,t\right)+c_{12}\sin\left(\sqrt{\frac{3k}{m}}\,t\right) \\ c_{21}\cos\left(\sqrt{\frac{1k}{m}}\,t\right)+c_{22}\sin\left(\sqrt{\frac{1k}{m}}\,t\right) \end{pmatrix}$$

Multiplying the matrices on the right side together and equating components, we get the following solutions to the differential equation that models the spring-weight system.

$$x_1 = c_{11}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) + c_{12}\sin\left(\sqrt{\frac{3k}{m}}\,t\right) + c_{21}\cos\left(\sqrt{\frac{1k}{m}}\,t\right) + c_{22}\sin\left(\sqrt{\frac{1k}{m}}\,t\right)$$

$$x_2 = -c_{11}\cos\left(\sqrt{\frac{3k}{m}}\,t\right) - c_{12}\sin\left(\sqrt{\frac{3k}{m}}\,t\right) + c_{21}\cos\left(\sqrt{\frac{1k}{m}}\,t\right) + c_{22}\sin\left(\sqrt{\frac{1k}{m}}\,t\right)$$

Exercise 2.3

Since we have n variables $x_1, x_2, \ldots, x_n$, we have n equations of motion

which are

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = -\frac{\partial V}{\partial x_1},\quad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = -\frac{\partial V}{\partial x_2},\quad \ldots,\quad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_n}\right] = -\frac{\partial V}{\partial x_n}.$$

The easiest way to construct these equations is to find each component. To find the left side of each of the equations of motion, we first differentiate the equation for kinetic energy with respect to $\dot{x}_i$ $(i=1, 2, \ldots, n)$.

$$\frac{\partial T}{\partial \dot{x}_1} = \frac{1}{2} m\left[2\dot{x}_1 + 0 + \ldots + 0\right] = m\dot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{2} m\left[0 + 2\dot{x}_2 + 0 + \ldots + 0\right] = m\dot{x}_2$$

$$\vdots$$

$$\frac{\partial T}{\partial \dot{x}_n} = \frac{1}{2} m\left[0 + \ldots + 0 + 2\dot{x}_n\right] = m\dot{x}_n$$

When we differentiate each of these with respect to time, we have

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = \frac{d}{dt}\left[m\dot{x}_1\right] = m\ddot{x}_1$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = \frac{d}{dt}\left[m\dot{x}_2\right] = m\ddot{x}_2$$

$$\vdots$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_n}\right] = \frac{d}{dt}\left[m\dot{x}_n\right] = m\ddot{x}_n .$$

The right side of each of the equations of motion is

$$-\frac{\partial V}{\partial x_1} = -\frac{1}{2}k\left[2b_{11}x_1 + 2b_{12}x_2 + \ldots + 2b_{1n}x_n\right] = -k\left[b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n\right]$$

$$-\frac{\partial V}{\partial x_2} = -\frac{1}{2}k\left[2b_{12}x_1 + 2b_{22}x_2 + \ldots + 2b_{2n}x_n\right] = -k\left[b_{12}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n\right]$$

$$\vdots$$

$$-\frac{\partial V}{\partial x_n} = -\frac{1}{2}k\left[2b_{1n}x_1 + 2b_{2n}x_2 + \ldots + 2b_{nn}x_n\right] = -k\left[b_{1n}x_1 + b_{2n}x_2 + \ldots + b_{nn}x_n\right]$$

If we combine these components, the equations of motion become

$$\ddot{x}_1 = \frac{k}{m}\left[-b_{11}x_1 - b_{12}x_2 - \ldots - b_{1n}x_n\right]$$

$$\ddot{x}_2 = \frac{k}{m}\left[-b_{12}x_1 - b_{22}x_2 - \ldots - b_{2n}x_n\right]$$

$$\vdots$$

$$\ddot{x}_n = \frac{k}{m}\left[-b_{1n}x_1 - b_{2n}x_2 - \ldots - b_{nn}x_n\right].$$

The equations of motion can be rewritten in matrix form as

$$\ddot{\vec{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -b_{11} & -b_{12} & \ldots & -b_{1n} \\ -b_{12} & -b_{22} & \ldots & -b_{2n} \\ \vdots & & & \vdots \\ -b_{1n} & -b_{2n} & \ldots & -b_{nn} \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{pmatrix} = \frac{k}{m} B \vec{X},$$

where B is an $n \times n$ symmetric matrix. Now we have an equation that should look very familiar to us.

$$\ddot{\vec{X}} = \frac{k}{m} B \vec{X}$$

Since B is a symmetric $n \times n$ matrix, there exists an orthogonal matrix P such that $P^{-1}BP=D$. The matrix D is the diagonal matrix whose entries along the main diagonal consist of the eigenvalues $\lambda_1, \dots, \lambda_n$ of B and the columns of P are the corresponding eigenvectors $\vec{X}_{\lambda_1}, \dots, \vec{X}_{\lambda_n}$ associated with the eigenvalues $\lambda_1, \dots, \lambda_n$, respectively. The following theory will be very similar to the theory that we developed for the spring-weight system with two blocks, except the sizes of the matrices and vectors will be $n \times n$ and $n \times 1$, respectively. If we multiply both sides of this equation by P^{-1} and use the identity $P\,P^{-1} = I_n$, we get

$$P^{-1}\ddot{\vec{X}} = P^{-1}\frac{k}{m} B \vec{X} = \frac{k}{m} P^{-1} B \left(P\,P^{-1}\right) \vec{X} = \frac{k}{m}\left(P^{-1} B P\right)\left(P^{-1}\vec{X}\right).$$

To simplify this equation, we let $\vec{U} = P^{-1}\vec{X}$. To introduce the vector variable $\vec{U}$, we substitute $\ddot{\vec{U}} = P^{-1}\ddot{\vec{X}}$ into this equation. We then substitute $P^{-1}BP = D$ and $\vec{U} = P^{-1}\vec{X}$ into the right side to obtain $\ddot{\vec{U}} = \frac{k}{m} D \vec{U}$. Now, multiplying both sides of this matrix equation by the matrix P, we get $P\ddot{\vec{U}} = \frac{k}{m} P D \vec{U}$. Rewriting the left side of $P\ddot{\vec{U}} = \frac{k}{m} P D \vec{U}$ gives

$$P\ddot{\vec{U}} = \left(P^{(1)}\,P^{(2)}\ldots P^{(n)}\right)\begin{pmatrix}\ddot{u}_1\\ \ddot{u}_2\\ \vdots\\ \ddot{u}_n\end{pmatrix} = P^{(1)}\ddot{u}_1 + P^{(2)}\ddot{u}_2 + \ldots + P^{(n)}\ddot{u}_n \;,$$

where $P^{(i)}$ represents the ith column $(i = 1, 2, \ldots, n)$ of the matrix P. We rewrite the right side to obtain

$$\frac{k}{m}PD\vec{U} = \frac{k}{m}\left[P^{(1)}\lambda_1 u_1 + P^{(2)}\lambda_2 u_2 + \ldots + P^{(n)}\lambda_n u_n\right].$$

Thus, $P\ddot{\vec{U}} = \frac{k}{m}PD\vec{U}$ can be written as

$$P^{(1)}\ddot{u}_1 + P^{(2)}\ddot{u}_2 + \ldots + P^{(n)}\ddot{u}_n = \frac{k}{m}\left[P^{(1)}\lambda_1 u_1 + P^{(2)}\lambda_2 u_2 + \ldots + P^{(n)}\lambda_n u_n\right].$$

We can simplify this equation by multiplying through by $\frac{k}{m}$, gathering terms, moving everything to the left side, and factoring out $P^{(1)}, P^{(2)}, \ldots,$ $P^{(n)}$ from each quantity.

$$\left[\ddot{u}_1 - \frac{k}{m}\lambda_1 u_1\right]P^{(1)} + \left[\ddot{u}_2 - \frac{k}{m}\lambda_2 u_2\right]P^{(2)} + \ldots + \left[\ddot{u}_n - \frac{k}{m}\lambda_n u_n\right]P^{(n)} = \vec{0}\;.$$

Since the columns of P are orthogonal, we know they are linearly independent, thus, the coefficients of the vectors $P^{(1)}, P^{(2)}, \ldots, P^{(n)}$ must be zero. If we set each of the coefficients in the above equation equal to zero, we have

$$\ddot{u}_1 - \frac{k}{m}\lambda_1 u_1 = 0, \quad \ddot{u}_2 - \frac{k}{m}\lambda_2 u_2 = 0, \quad \ldots \quad , \ddot{u}_n - \frac{k}{m}\lambda_n u_n = 0 .$$

These are all second order linear differential equations which can be solved using basic techniques. If we let $r_i = -\frac{k}{m}\lambda_i$, where $i=1, 2, \ldots, n$, then these equations become

$$\ddot{u}_1 + r_1 u_1 = 0, \quad \ddot{u}_2 + r_2 u_2 = 0, \quad \ldots \quad , \ddot{u}_n + r_n u_n = 0 .$$

Using the following formulas, we can solve for the vector $\vec{U}$. (Note: To determine whether r_i is zero, negative or positive, substitute λ_i into

$r_i = -\frac{k}{m}\lambda_i$.)

If $r_i = 0$, then $u_i = c_{i1}t + c_{i2}$

If $r_i < 0$, then $u_i = c_{i1}e^{r_i t} + c_{i2}e^{-r_i t}$

If $r_i > 0$, then $u_i = c_{i1}\cos\left(\sqrt{r_i}\, t\right) + c_{i2}\sin\left(\sqrt{r_i}\, t\right)$.

The solution to the original system of differential equations $\ddot{\vec{X}} = \frac{k}{m} B \vec{X}$ is found by substituting the values for both the matrix P and the vector $\vec{U}$ into the equation $\vec{X} = P \vec{U}$.

Exercise 2.4

Since the spring-weight system lies free in the xy-plane, the entire system can move vertically up or down, horizontally to the left or right, or rotate. These types of motion are called rigid motions. The system can also vibrate producing the motions that are described by Figures 2.6 and 2.7.

Exercise 2.5

(a) Suppose the three masses are moved to the right causing the first three springs to stretch by different amounts and causing the fourth spring to be compressed. This is depicted in Figure B.1.

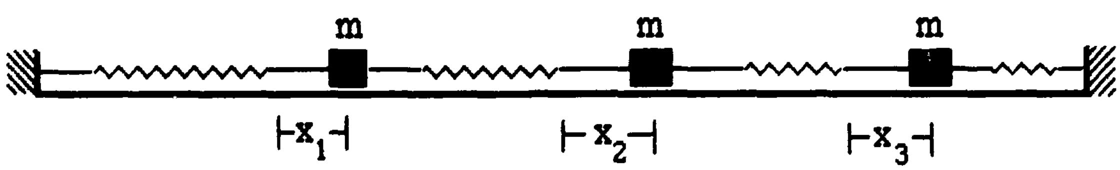


Figure B.1

First, we need the equation which describes the kinetic energy of the system in Figure B.1.

$$T = \frac{1}{2} m \left[\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right]$$

The potential energy of the system is the sum of the potential energies of each spring. Spring 1 is stretched from its equilibrium position by the amount x_1, so the potential energy for spring 1 is $V_1 = \frac{1}{2} k x_1^2$. Spring 2 is stretched from its equilibrium position by the amount $x_1 - x_2$, so that

$V_2 = \frac{1}{2} k \left(x_1 - x_2\right)^2$ is the potential energy for spring 2. Spring 3 is stretched from its equilibrium position by the amount $x_2 - x_3$, producing a potential

energy of $V_3 = \frac{1}{2} k \left(x_2 - x_3\right)^2$ for spring 3. Spring 4 is compressed from its equilibrium position by the amount x_3. Thus, the potential energy for spring 4 is $V_4 = \frac{1}{2} k x_3^2$. Therefore, the potential energy of the system is

$$V = \sum_{i=1}^{4} V_i = \frac{1}{2} k \left[x_1^2 + \left(x_1 - x_2\right)^2 + \left(x_2 - x_3\right)^2 + x_3^2 \right]$$

$$= k \left[x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 \right] .$$

Since we have three variables x_1, x_2, and x_3, we have three equations of motion which are

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = -\frac{\partial V}{\partial x_1}, \quad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = -\frac{\partial V}{\partial x_2} \quad \text{and} \quad \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_3}\right] = -\frac{\partial V}{\partial x_3} .$$

The easiest way to construct these equations is to find each component. To find the left side of each of the equations of motion, we first differentiate the equation for kinetic energy with respect to $\dot{x}_i$ (i=1,2,3).

$$\frac{\partial T}{\partial \dot{x}_1} = \frac{1}{2} m\left[2\dot{x}_1 + 0 + 0\right] = m\dot{x}_1$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{2} m\left[0 + 2\dot{x}_2 + 0\right] = m\dot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = \frac{1}{2} m\left[0 + 0 + 2\dot{x}_3\right] = m\dot{x}_3$$

Now, differentiating each of these with respect to time, we have

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = \frac{d}{dt}\left[m\dot{x}_1\right] = m\ddot{x}_1$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = \frac{d}{dt}\left[m\dot{x}_2\right] = m\ddot{x}_2$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_3}\right] = \frac{d}{dt}\left[m\dot{x}_3\right] = m\ddot{x}_3$$

The right side of each of the equations of motion is

$$-\frac{\partial V}{\partial x_1} = -k\left[2x_1 - x_2\right] = k\left[-2x_1 + x_2\right]$$

$$-\frac{\partial V}{\partial x_2} = -k\left[2x_2 - x_1 - x_3\right] = k\left[x_1 - 2x_2 + x_3\right]$$

$$-\frac{\partial V}{\partial x_3} = -k\left[2x_3 - x_2\right] = k\left[x_2 - 2x_3\right].$$

If we equate these components, the equations of motion become

$$\ddot{x}_1 = \frac{k}{m}\left[-2x_1 + x_2\right]$$

$$\ddot{x}_2 = \frac{k}{m}\left[x_1 - 2x_2 + x_3\right]$$

$$\ddot{x}_3 = \frac{k}{m}\left[x_2 - 2x_3\right].$$

The equations of motion can be rewritten in matrix form, which is the system of differential equations modeling the spring-weight system in Figure B.1.

$$\ddot{\vec{X}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m} A \vec{X}$$

(b) Since A is a symmetric matrix, all of its eigenvalues are real and A is diagonalizable. We begin by finding the eigenvalues of the matrix A.

$$\det(A-\lambda I)=\det\begin{pmatrix}-2-\lambda & 1 & 0\\ 1 & -2-\lambda & 1\\ 0 & 1 & -2-\lambda\end{pmatrix}=(\lambda+2)(\lambda+2-\sqrt{2})(\lambda+2+\sqrt{2})$$

If we set $\det(A-\lambda I)$ equal to zero and solve for λ, we find the eigenvalues are $\lambda_1=-2$, $\lambda_2=-2+\sqrt{2}$ and $\lambda_3=-2-\sqrt{2}$. Thus, there exists an invertible matrix P such that $P^{-1}AP=D$. D is the diagonal matrix whose entries along the main diagonal consist of the eigenvalues of A and the columns of P are the corresponding eigenvectors. To find P, we need to find the eigenvectors associated with $\lambda_1=-2$, $\lambda_2=-2+\sqrt{2}$ and $\lambda_3=-2-\sqrt{2}$. For $\lambda_1=-2$ we have to reduce the following augmented matrix

$$\left(\begin{array}{ccc|c}-2+2 & 1 & 0 & 0\\ 1 & -2+2 & 1 & 0\\ 0 & 1 & -2+2 & 0\end{array}\right).$$

to obtain $\left(\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$, which yields the following equations: $x_1=-x_3$ and $x_2=0$. If we let $x_1=-1$, an eigenvector associated with $\lambda_1=-2$ is

$$\vec{X}_{\lambda_1=-2}=\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}.$$

For $\lambda_2=-2+\sqrt{2}$ we have to reduce the following augmented matrix.

$$\left(\begin{array}{ccc|c} -2-(-2+\sqrt{2}) & 1 & 0 & 0 \\ 1 & -2-(-2+\sqrt{2}) & 1 & 0 \\ 0 & 1 & -2-(-2+\sqrt{2}) & 0 \end{array}\right),$$

to obtain $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, which yields the following equations: $x_1=x_3$

and $x_2=\sqrt{2}x_3$. If we let $x_3=1$, an eigenvector associated with $\lambda_1=-2+\sqrt{2}$ is

$$\vec{X}_{\lambda_2=-2+\sqrt{2}} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

For $\lambda_2=-2-\sqrt{2}$ we have to reduce the following augmented matrix.

$$\left(\begin{array}{ccc|c} -2-(-2-\sqrt{2}) & 1 & 0 & 0 \\ 1 & -2-(-2-\sqrt{2}) & 1 & 0 \\ 0 & 1 & -2-(-2-\sqrt{2}) & 0 \end{array}\right)$$

to obtain $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, which yields the following equations: $x_1=x_3$

and $x_2=-\sqrt{2}x_3$. If we let $x_3=1$, an eigenvector associated with $\lambda_1=-2-\sqrt{2}$ is

$\vec{X}_{\lambda_3=-2-\sqrt{2}} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$. Using the theory that we developed in Exercise 2.3, we know that we must first solve the differential equation $P\ddot{\vec{U}} = \frac{k}{m} P D \vec{U}$, which leads to

$$\left[\ddot{u}_1 - \frac{k}{m}\lambda_1 u_1\right] P^{(1)} + \left[\ddot{u}_2 - \frac{k}{m}\lambda_2 u_2\right] P^{(2)} + \left[\ddot{u}_3 - \frac{k}{m}\lambda_3 u_3\right] P^{(3)} = \vec{0}.$$

Since the columns of P are orthogonal, we know they are linearly independent. Thus, the coefficients of the vectors $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$, must be zero. If we set each of the coefficients in the above equation equal to zero, we have

$$\ddot{u}_1 - \frac{k}{m}\lambda_1 u_1 = 0, \qquad \ddot{u}_2 - \frac{k}{m}\lambda_2 u_2 = 0, \qquad \ddot{u}_3 - \frac{k}{m}\lambda_3 u_3 = 0.$$

Substituting in the eigenvalues, we obtain

$$\ddot{u}_1 - \frac{k}{m}(-2)\, u_1 = 0, \;\; \ddot{u}_2 - \frac{k}{m}(-2+\sqrt{2})\, u_2 = 0, \;\; \ddot{u}_3 - \frac{k}{m}(-2-\sqrt{2})\, u_3 = 0.$$

After simplifying these equations they become

$$\ddot{u}_1 + \frac{2k}{m} u_1 = 0, \;\; \ddot{u}_2 + \frac{(2-\sqrt{2})k}{m} u_2 = 0, \;\; \ddot{u}_3 + \frac{(2+\sqrt{2})k}{m} u_3 = 0.$$

From these differential equations we observe that

$$r_1 = \frac{2k}{m}, \qquad r_2 = \frac{(2-\sqrt{2})k}{m}, \qquad r_3 = \frac{(2+\sqrt{2})k}{m}.$$

Since each r_i $(i=1, 2, 3)$ is greater than zero, the solutions to these second order linear differential equations are

$$u_1 = c_{11}\cos\left(\sqrt{\frac{2k}{m}}\, t\right) + c_{12}\sin\left(\sqrt{\frac{2k}{m}}\, t\right)$$

$$u_2 = c_{21}\cos\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right) + c_{22}\sin\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right)$$

$$u_3 = c_{31}\cos\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right) + c_{32}\sin\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right).$$

The solution to the original system of differential equations $\ddot{\vec{X}} = \frac{k}{m} A \vec{X}$ is found by substituting the values for both the matrix P and the vector $\vec{U}$ into the equation $\vec{X} = P\,\vec{U}$.

$$\vec{X} = P\,\vec{U}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_{11}\cos\left(\sqrt{\frac{2k}{m}}\, t\right) + c_{12}\sin\left(\sqrt{\frac{2k}{m}}\, t\right) \\ c_{21}\cos\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right) + c_{22}\sin\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right) \\ c_{31}\cos\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right) + c_{32}\sin\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right) \end{pmatrix}$$

Multiplying the matrices on the right side together and equating components, we get the following solutions to the differential equation which model the spring-weight system.

$$x_1 = -c_{11}\cos\left(\sqrt{\frac{2k}{m}}\, t\right) - c_{12}\sin\left(\sqrt{\frac{2k}{m}}\, t\right)$$

$$+ c_{21}\cos\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right) + c_{22}\sin\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right)$$

$$+ c_{31}\cos\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right) + c_{32}\sin\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right)$$

$$x_2 = \sqrt{2}\, c_{21}\cos\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right) + \sqrt{2}\, c_{22}\sin\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right)$$

$$- \sqrt{2}\, c_{31}\cos\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right) - \sqrt{2}\, c_{32}\sin\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right)$$

$$x_3 = c_{11}\cos\left(\sqrt{\frac{2k}{m}}\, t\right) + c_{12}\sin\left(\sqrt{\frac{2k}{m}}\, t\right)$$

$$+ c_{21}\cos\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right) + c_{22}\sin\left(\sqrt{\frac{(2-\sqrt{2})k}{m}}\, t\right)$$

$$+ c_{31}\cos\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right) + c_{32}\sin\left(\sqrt{\frac{(2+\sqrt{2})k}{m}}\, t\right)$$

(c) To describe the configurations in which the spring-weight system vibrates, we need to write each u_i in terms of the x_i. This can be done by using the matrix equation $\vec{X} = P\vec{U}$ or $\vec{U} = P^{-1}\vec{X}$. We first find P^{-1} and substitute it into the matrix equation.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Multiplying these matrices together and equating components, we get the following three equations.

$$u_1 = \frac{-x_1 + x_3}{2}$$

$$u_2 = \frac{x_1 + \sqrt{2}\, x_2 + x_3}{4}$$

$$u_3 = \frac{x_1 - \sqrt{2}\, x_2 + x_3}{4}$$

These three equations describe the relationships between the variables x_1, x_2 and x_3. The three modes in which this spring-weight system vibrates are $u_1 = \frac{-x_1 + x_3}{2}$ with $u_2=0$ and $u_3=0$, $u_2 = \frac{x_1 + \sqrt{2}\, x_2 + x_3}{4}$ with $u_1=0$ and $u_3=0$, and $u_3 = \frac{x_1 - \sqrt{2}\, x_2 + x_3}{4}$ with $u_1=0$ and $u_2=0$.

In the first mode, $u_1 = \frac{-x_1 + x_3}{2}$ with $u_2 = 0$ and $u_3 = 0$. Since $\frac{-x_1 + x_3}{2}$ represents how the distance between blocks one and three is changing, the first mode of vibration describes how the distance between the two blocks is changing. We visualize this by considering a series of diagrams similar to those in Figure 2.6. Recall, the banner is made of an elastic material and indicates the distance between the two blocks. When this system vibrates, we see the banner contracting and stretching with a frequency associated with λ_1. This is indicated by the series of "snapshots" of the spring-weight system in motion seen in Figure B.2.

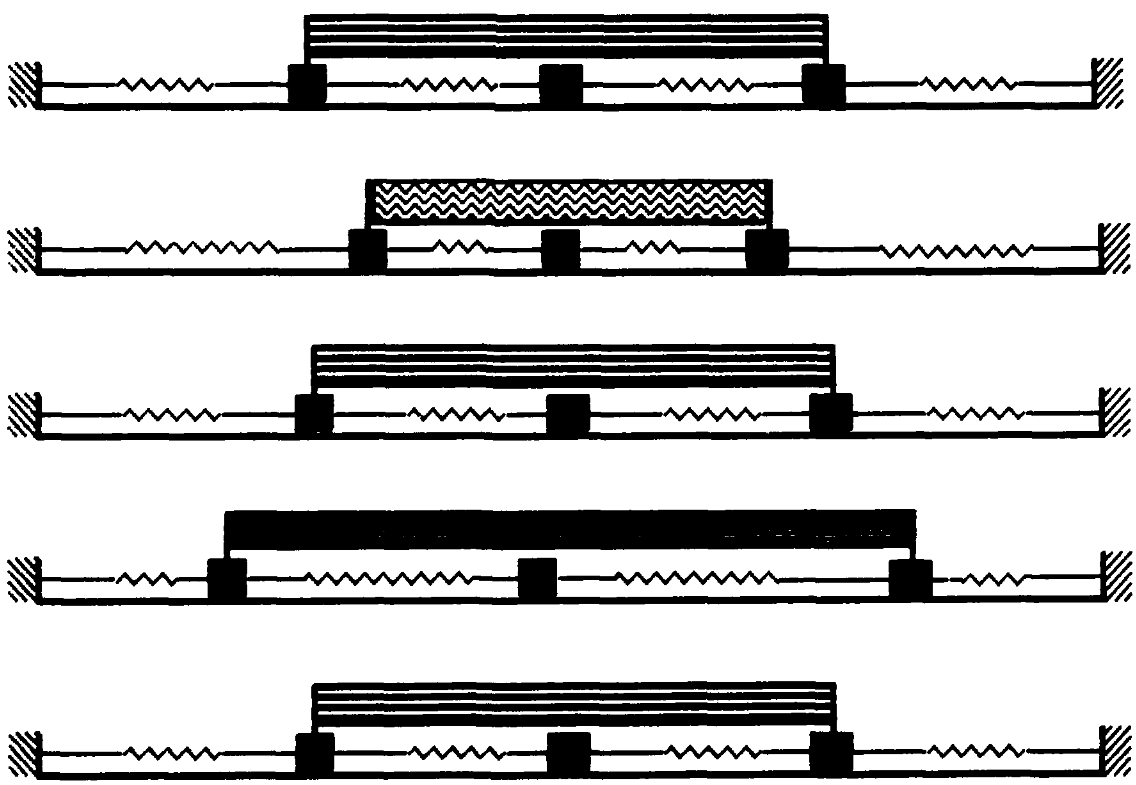


Figure B.2

In the second mode, $u_2 = \frac{x_1 + \sqrt{2}\, x_2 + x_3}{4}$ with $u_1 = 0$ and $u_3 = 0$. To visualize this mode of vibration, we consider a series of diagrams similar to those in Figure 2.6. When this system vibrates, we see the flag moving back and forth with a frequency associated with λ_2. This is indicated by the series of "snapshots" of the spring-weight system in motion as seen in Figure B.3.

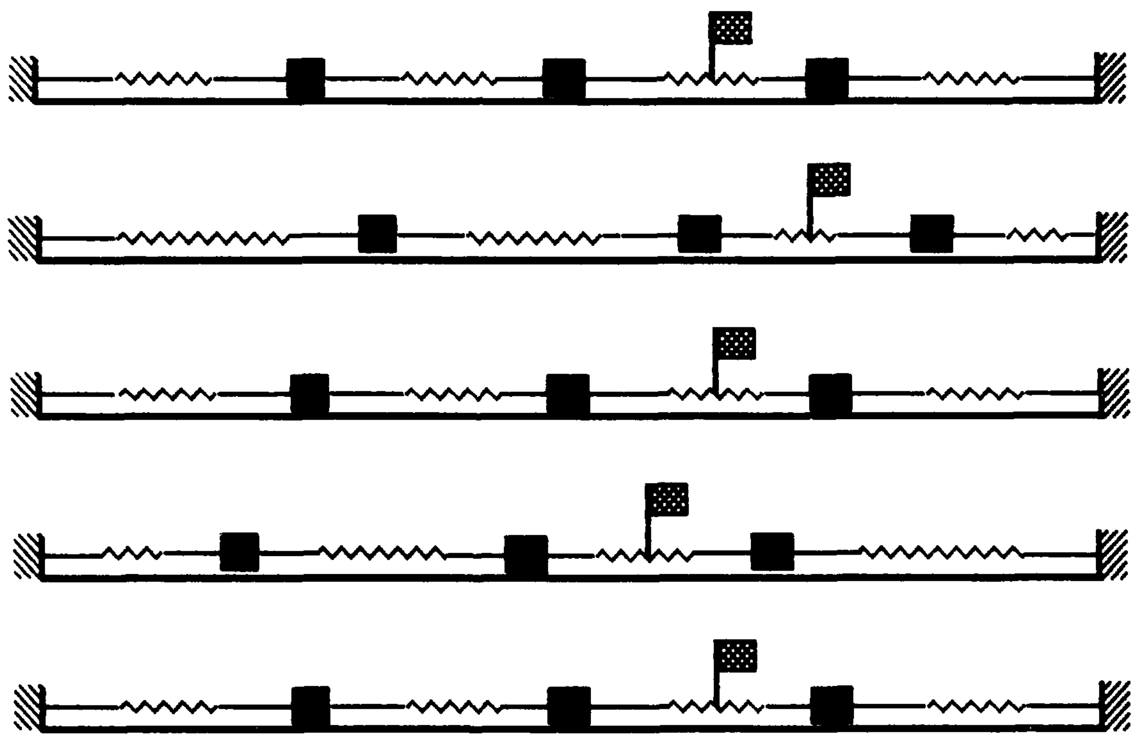


Figure B.3

In the third mode, $u_3 = \frac{x_1 - \sqrt{2}\,x_2 + x_3}{4}$ with $u_1 = 0$ and $u_2 = 0$. To visualize this mode of vibration, we consider a series of diagrams similar to those in Figure 2.6. When this system vibrates, we see the flag moving back and forth with a frequency associated with λ_3. This is indicated by the series of "snapshots" of the spring-weight system in motion, as seen in Figure B.4.

Figure B.4

Exercise 2.6

Since the spring-weight system lies in the xy-plane, the entire system can move vertically up or down, horizontally to the left or right, or rotate. Besides these rigid motions, the system can also vibrate producing the motion that were described in Exercise 2.6.

Exercise 3.1

We write $f\left(\frac{1}{L}\right)$ as a Taylor series expansion expanded about zero.

$$f\left(\frac{1}{L}\right) = f(0) + \frac{f'(0)}{1!}\frac{1}{L} + \frac{f''(0)}{2!}\left(\frac{1}{L}\right)^2 + \ldots$$

We use $f\left(\frac{1}{L}\right)$, which has been repeated below for convenience, to find f(0), f'(0) and f''(0).

$$f\left(\frac{1}{L}\right) = \left\{1 + \frac{1}{L}\left(\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right) + \frac{1}{L^2}\left(\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2\right)\right\}^{\frac{1}{2}}$$

$$f(0) = 1$$

$$f'(0) = \frac{1}{2}\left[\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\right]$$

$$f''(0) = -\frac{1}{4}\left[\left(x_1 - x_2\right) - \left(y_1 - y_2\right)\right]^2 + \left(x_1 - x_2\right)^2 - \left(y_1 - y_2\right)^2$$

Substituting these into the Taylor series expansion and simplifying, we have

$$f\left(\frac{1}{L}\right) = 1 + \frac{\frac{1}{2}\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)}{1!}\frac{1}{L}$$

$$+ \frac{-\frac{1}{4}\left[\left(x_1 - x_2\right) - \left(y_1 - y_2\right)\right]^2 + \left(x_1 - x_2\right)^2 - \left(y_1 - y_2\right)^2}{2!}\frac{1}{L^2} + \ldots$$

$$= 1 + \frac{1}{2}\left(x_1 - x_2\right) - \sqrt{3}\left(y_1 - y_2\right)\frac{1}{L} + \text{higher power terms.}$$

Exercise 3.2

Figure B.5 contains the coordinates of the blocks and the distance formula for the spring from Figure 3.3.

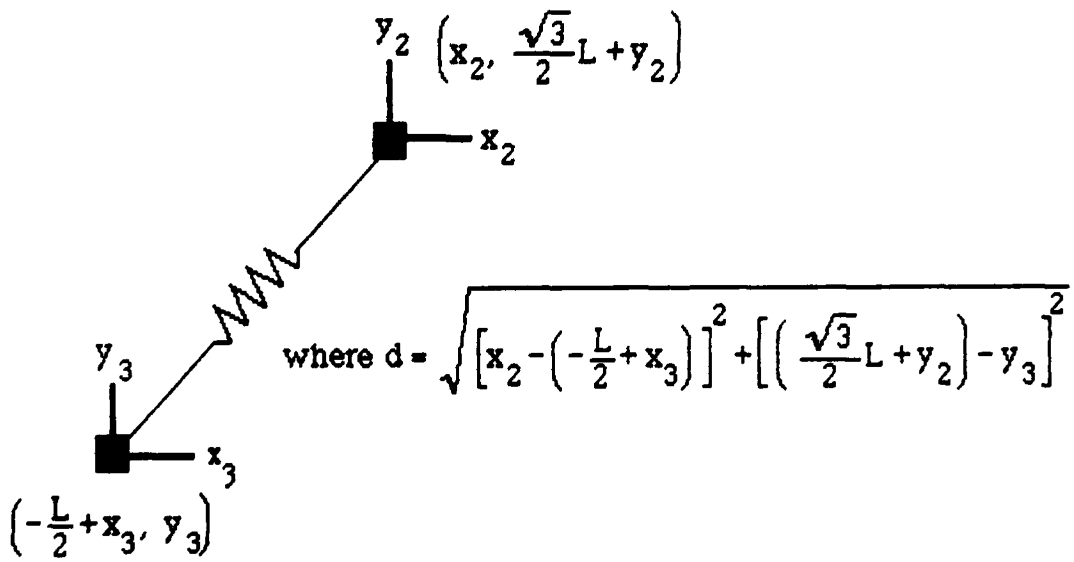


Figure B.5

The potential energy of this spring is one half the spring constant k times the square of the distance that the spring is stretched. The displacement of the spring from its equilibrium position (the distance that the spring is stretched) is $|d-L|$. Expressing the potential energy of the spring in terms of $|d-L|$, we have

$$V_{23} = \frac{1}{2} k \, |d - L|^2 .$$

We want to rewrite V_{23} using the variables x_2, y_2, x_3 and y_3. To do this, we must first simplify the expression for the distance d.

$$d = \left\{ \left[x_2 - \left(-\frac{L}{2} + x_3 \right) \right]^2 + \left[\left(\frac{\sqrt{3}}{2} L + y_2 \right) - y_3 \right]^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ \left[\frac{L}{2} + \left(x_2 - x_3 \right) \right]^2 + \left[\frac{\sqrt{3}}{2} L + \left(y_2 - y_3 \right) \right]^2 \right\}^{\frac{1}{2}}$$

$$= L \left\{ 1 + \frac{1}{L} \left[\left(x_2 - x_3 \right) + \sqrt{3} \left(y_2 - y_3 \right) \right] + \frac{1}{L^2} \left[\left(x_2 - x_3 \right)^2 + \left(y_2 - y_3 \right)^2 \right] \right\}^{\frac{1}{2}}$$

We can rewrite the quantity expressed by the square root using its Taylor series expansion. The terms of higher powers have been grouped together for convenience.

$$d \approx L \left\{ 1 + \frac{\frac{1}{2} \left[\left(x_2 - x_3 \right) + \sqrt{3} \left(y_2 - y_3 \right) \right]}{1!} \frac{1}{L} + \text{terms of higher powers} \right\}$$

This expression can be simplified by multiplying through by L and then moving the L to the left side. The resulting quantity is what we wanted to find.

$$d - L = \frac{1}{2}\left[\left(x_2 - x_3\right) + \sqrt{3}\left(y_2 - y_3\right)\right] + \text{terms of higher powers}$$

This quantity can now be substituted into the formula for the potential energy V_{23}.

$$V_{23} = \frac{1}{2}k \left| d - L \right|^2$$

$$= \frac{k}{2}\left\{ \frac{1}{4}\left[\left(x_2 - x_3\right) + \sqrt{3}\left(y_2 - y_3\right)\right]^2 + \text{terms of higher powers}\right\}$$

The Taylor series expansion for potential energy can not have nonzero linear terms, since the system has an equilibrium configuration. Also, we are considering only small vibrations so we ignore the terms of higher powers. The formula for potential energy V_{23}, is

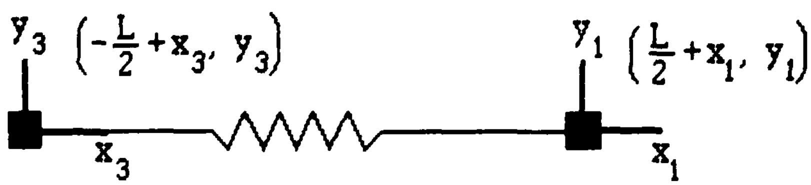
$$V_{23} = \frac{k}{2}\left\{\frac{1}{4}\left[\left(x_2 - x_3\right) + \sqrt{3}\left(y_2 - y_3\right)\right]^2\right\}$$

$$= \frac{k}{2}\left\{\frac{x_2^2}{4} + \frac{x_3^2}{4} + \frac{3y_2^2}{4} + \frac{3y_3^2}{4} - \frac{1}{2}x_2x_3 + \frac{\sqrt{3}}{2}x_2y_2 - \frac{\sqrt{3}}{2}x_3y_2\right.$$

$$\left. - \frac{\sqrt{3}}{2}x_2y_3 + \frac{\sqrt{3}}{2}x_3y_3 - \frac{3}{2}y_2y_3\right\}$$

Exercise 3.3

Figure B.6 contains the coordinates of the blocks and the distance formula for the spring from Figure 3.4.



$$\text{where } d = \sqrt{\left[\left(\frac{L}{2}+x_1\right) - \left(-\frac{L}{2}+x_3\right)\right]^2 + \left[y_1 - y_3\right]^2}$$

Figure B.6

The potential energy of this spring is one half the spring constant k, times the square of the distance that the spring is stretched. The displacement of the spring from its equilibrium position (the distance that the spring is stretched) is $|d-L|$. Expressing the potential energy of the spring in terms of $|d-L|$, we have

$$V_{13} = \frac{1}{2} k \left| d - L \right|^2 .$$

We want to rewrite V_{13} using the variables x_1, y_1, x_3 and y_3. To do this we must first simplify the expression for the distance d.

$$d = \left\{ \left[\left(\frac{L}{2} + x_1\right) - \left(-\frac{L}{2} + x_3\right) \right]^2 + \left[y_1 - y_3 \right]^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ L^2 + 2L\left(x_1 - x_3\right) + \left(x_1 - x_3\right)^2 + \left(y_1 - y_3\right)^2 \right\}^{\frac{1}{2}}$$

$$= L \left\{ 1 + \frac{2}{L}\left(x_1 - x_3\right) + \frac{1}{L^2}\left[\left(x_1 - x_3\right)^2 + \left(y_1 - y_3\right)^2 \right] \right\}^{\frac{1}{2}}$$

We can rewrite the quantity expressed by the square root using its Taylor series expansion. The terms of higher powers have been grouped together for convenience.

$$d = L \left\{ 1 + \frac{\frac{1}{2}\left[2\left(x_1 - x_3\right)\right]}{1!} \frac{1}{L} + \text{higher power terms} \right\}$$

This expression can be simplified by multiplying through by L and then moving the L to the left side. The resulting quantity is what we wanted to find.

$$d - L = \left(x_1 - x_3\right) + \text{higher power terms}$$

This quantity can now be substituted into the formula for the potential energy V_{13}.

$$V_{13} = \frac{1}{2} k \left| d - L \right|^2$$

$$= \frac{k}{2} \left\{ \left(x_1 - x_3\right)^2 + \text{higher power terms} \right\}$$

The Taylor series expansion for potential energy can not have any nonzero linear terms, since the system has an equilibrium configuration. Also, we are considering only small vibrations so we ignore the terms of higher powers. The formula for potential energy V_{13}, is

$$V_{13} = \frac{k}{2} \left(x_1 - x_3\right)^2$$

$$= \frac{k}{2} \left\{ x_1^2 + x_3^2 - 2 x_1 x_3 \right\}$$

Exercise 3.4

Substituting $\vec{X}_{\lambda_j=0} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ into $V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}^T_{\lambda_j=0} \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$, gives

$$V\left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] = (1\ 1\ 1\ 0\ 0\ 0)\ \frac{-k}{8m} A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{-k}{8m}(1\ 1\ 1\ 0\ 0\ 0)\begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{-k}{8m}(0\ 0\ 0\ 0\ 0\ 0)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

Exercise 3.5

Substituting $\vec{X}_{\lambda_2=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ into $V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}^T_{\lambda_j=0} \dfrac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$, gives

$$V\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right] = (0\ 0\ 0\ 1\ 1\ 1)\,\frac{-k}{8m} A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{-k}{8m}(0\ 0\ 0\ 1\ 1\ 1)\begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{-k}{8m}(0\ 0\ 0\ 0\ 0\ 0)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

Exercise 3.6

Substituting $\vec{X}_{\lambda_3=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$ into $V\left(\vec{X}_{\lambda_j=0}\right) = \vec{X}^T_{\lambda_j=0} \frac{-k}{8m} A \vec{X}_{\lambda_j=0} = 0$,

gives

$$V\left[\begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}\right] = (-\sqrt{3}, 2\sqrt{3}, -\sqrt{3}, -3, 0, 3) \frac{-k}{8m} A \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix}$$

$$= \frac{-k}{8m}(-\sqrt{3}, 2\sqrt{3}, -\sqrt{3}, -3, 0, 3) \begin{pmatrix} -5 & 1 & 4 & \sqrt{3} & -\sqrt{3} & 0 \\ 1 & -2 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 4 & 1 & -5 & 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -\sqrt{3} & 0 & -3 & 3 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 3 & -6 & 3 \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 3 & -3 \end{pmatrix} \vec{X}_{\lambda_3}$$

$$= \frac{-k}{8m}(0\ 0\ 0\ 0\ 0\ 0) \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix} = 0$$

Now, show that $\vec{X}_{\lambda_3=0}$ is orthogonal to both $\vec{X}_{\lambda_1=0}$ and $\vec{X}_{\lambda_2=0}$.

$$\vec{X}_{\lambda_3=0} \bullet \vec{X}_{\lambda_1=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\vec{X}_{\lambda_3=0} \bullet \vec{X}_{\lambda_2=0} = \begin{pmatrix} -\sqrt{3} \\ 2\sqrt{3} \\ -\sqrt{3} \\ -3 \\ 0 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

<u>Exercise 3.7</u>

To show that $\vec{X}_{\lambda_1=-6}$ and $\vec{X}_{\lambda_2=-6}$ are orthogonal, we show their dot product is zero.

$$\vec{X}_{\lambda_1=-6} \bullet \vec{X}_{\lambda_2=-6} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} \bullet \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ -1 \end{pmatrix} = 0$$